Discrete MRF Inference of Marginal Densities for Non-uniformly Discretized Variable Space

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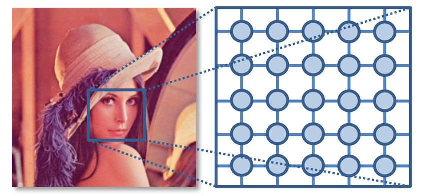
- We present a new approach to convert continuous MRFs to discrete MRFs, which enables non-uniform discretization of the variable space
- It yields extended MF/BP algorithms, which can achieve better trade-off between estimation accuracy and computational complexity
- We also propose a "dynamic discretization method" that adaptively discretizes the variable space depending on the necessity

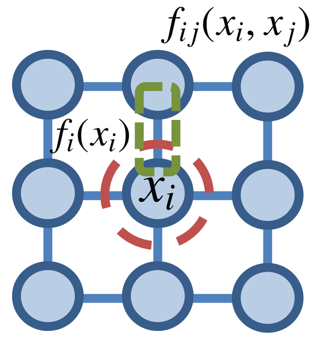
Introduction

MRF models:

$$Q(\mathbf{x}) = \frac{1}{Z}\exp(-E(\mathbf{x})) \qquad E(\mathbf{x}) = \sum_i f_i(x_i) + \sum_{(i,j)\in\mathcal{E}} f_{ij}(x_i, x_j)$$



Two types of inference used with MRFs:



MAP inference

$$\mathbf{x}^* = \arg\max_{\mathbf{x}} Q(\mathbf{x}) = \arg\min_{\mathbf{x}} E(\mathbf{x})$$

MPM inference

This study = Estimation of marginal densities

$$x_i^* = \arg\max_{x_i} q_i(x_i)$$

$$q_i(x_i) = \int dx_1 \cdots \int dx_{i-1} \int dx_{i+1} \cdots \int dx_N Q(\mathbf{x})$$

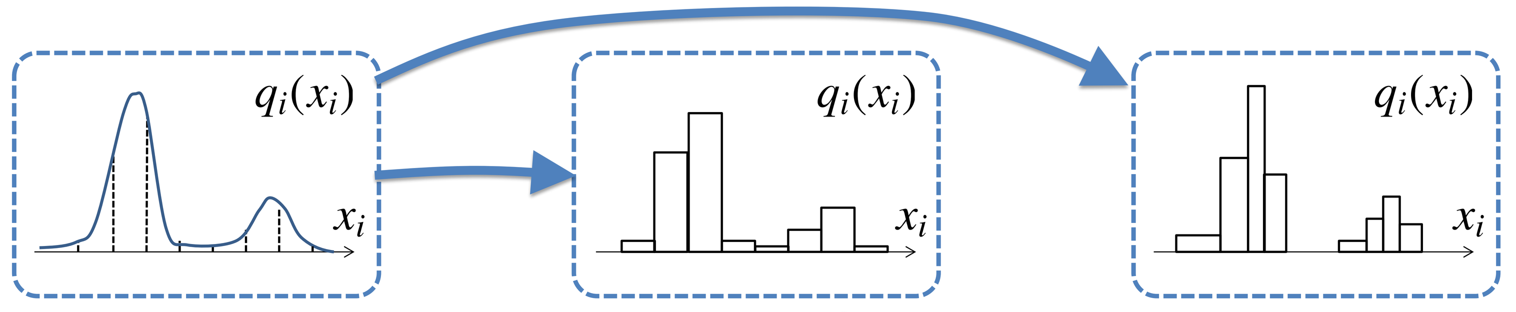
Continuous/Discrete MRFs

Because of the limitation with continuous MRFs, the estimation is often performed in the discrete domain even for continuous problems; the marginal densities are approximated by discrete densities

Tradeoff: accuracy vs. computational cost

Accurate approximation requires fine discretization, which will increase the computational cost; the variable space is uniformly discretized

Our idea: Discretize the variable space in a non-uniform manner such that the approximation will be accurate with a small number of discretization



Variational approach

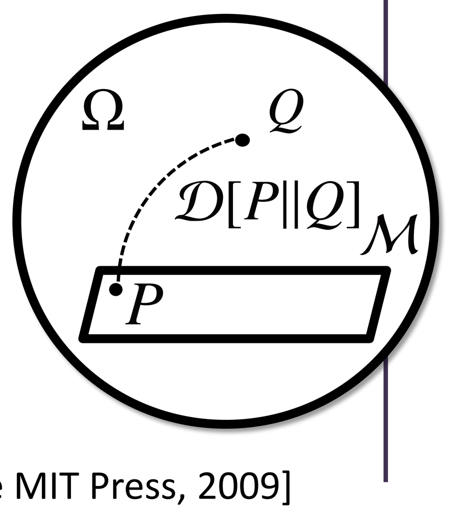
How to cope with the intractability of computing marginal densities:

1. Consider P such that its marginal densities will be easily calculated:

MF $P(\mathbf{x}) = \prod_i p_i(x_i)$ BP $P(\mathbf{x}) = \frac{\prod_{i,j} p_{ij}(x_i, x_j)}{\prod_i p_i(x_i)^{z_i - 1}}$

2. Find P minimizing the KL distance between P and Q.

$$\mathcal{D}[P\|Q] \to \min \Leftrightarrow F[P] = \langle E\rangle_P - S[P] \to \min$$



[Koller-Friedman, Probabilistic Graphical Models: Principles and Techniques, The MIT Press, 2009]

Our approach

Conventional way of converting continuous MRFs to discrete ones

1. Discretize the variable space
2. Redefine the energy in the discrete domain

→ *The original continuous energy (if any) is discarded*

Our approach:

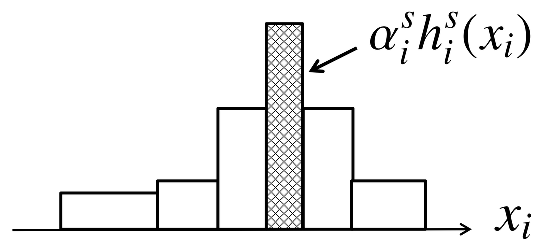
1. Represent the marginal density as a mixture of rectangular densities
2. Derive MF/BP message exchange rules in the variational framework

→ *The original continuous energy is maintained*

weight / rectangular density

$$p_i(x_i) = \sum_{s=1}^{S_i} \alpha_i^s h_i^s(x_i)$$

$$p_{ij}(x_i, x_j) = \sum_s \sum_t \alpha_{ij}^{st} h_i^s(x_i) h_j^t(x_j)$$



Results: New MF/BP algorithms

MF

$$\alpha_i^s \propto \exp\left[-\left(f_i^s - B_i^s\right) + \sum_{j\in N_i}\sum_{t=1}^{S_j} f_{ij}^{st}\alpha_j^t\right]$$

Term compensating the non-uniform rectangular widths:

$$B_i^s = -\int h_i^s(x_i)\ln h_i^s(x_i)dx_i$$

BP

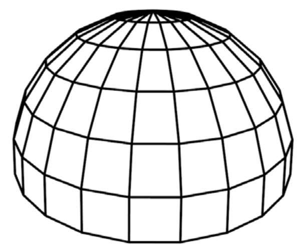
$$m_{ij}^t \leftarrow \sum_s \phi_i^s \psi_{ij}^{st} \prod_{k\in N_i\setminus j} m_{ki}^s$$

$$\alpha_i^s \propto \phi_i^s \prod_{k\in N_i} m_{ki}^s$$

$$\begin{cases} \phi_i^s = \exp\left[-(f_i^s + (z_i - 1)B_i^s)\right] \\ \psi_{ij}^{st} = \exp\left[-(f_{ij}^{st} - B_i^s - B_j^t)\right] \end{cases}$$

Advantages:

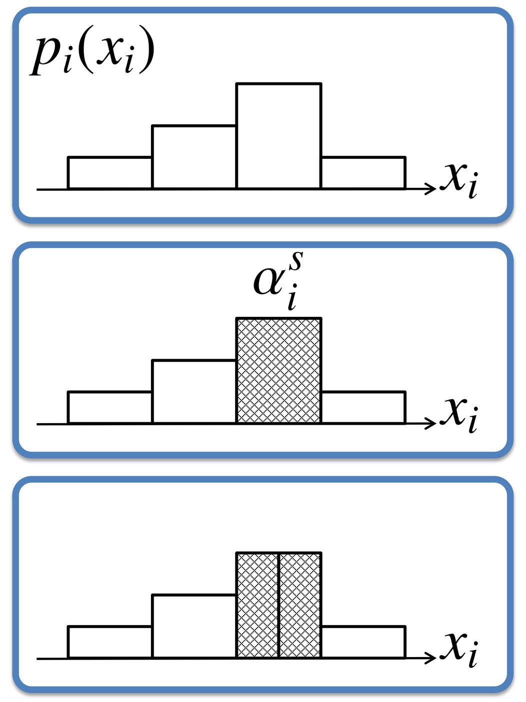
1. Enable non-uniform discretization
   → Better trade-off between accuracy and computational complexity
2. Could be useful for dealing with the variable spaces that are non-Euclidean and are difficult to discretize uniformly



Dynamic discretization of the variable space

- If a prior knowledge is available about where is important in the variable space, the application of our method is straightforward
- We propose a "*dynamic discretization method*" that can be used when no such knowledge is available

1. Discretize the variable space coarsely and estimate coarse marginal densities
2. Identify the block whose mixture weight α_i^s is the largest
3. Divide this block into subblocks by adjusting the values of the weights or the messages

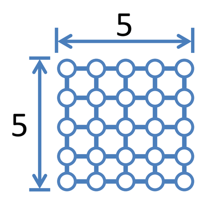


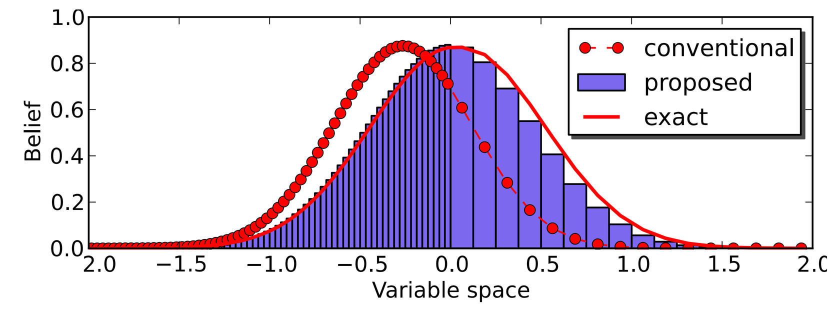
Experimental results (BP)

1. Effects of non-uniform discretization (Gaussian grid MRF)

- Neglecting the non-uniformity results in a biased estimates
- Our method gives a unbiased result

$$E(\mathbf{x}) = \sum_i x_i^2 + \sum_{i,j}(x_i - x_j)^2$$





2. Stereo Matching