3. Matrices and linear algebra I

- Accessing elements
- Basic operations
- Norms
- Inverse matrix
- Linear equation

Accessing elements

- As you have learned, ’ ; ’ indicates the end of a row; matrices of any size can be created in this way

```
>> A=[1,2,3;2,3,4]
A =
    1    2    3
    2    3    4
```

```
>> B=[1,2;2,3;3,4]
B =
    1    2
    2    3
    3    4
```

- Specify row and column indices to access an element
- A whole row or a whole column can be represented using ‘:’

```
>> A(2,3)
ans =  4
>> A(1,2)
ans =  2
```

```
>> B(3,:)
ans =
    3    4
```

```
>> B(:,1)
ans =
    1
    2
    3
```

Quick creation of several matrices by functions

- Identity matrix: eye (m)
- Matrix of all 1's: ones(m,n)
- Matrix of all 0's: zeros(m,n)

Remark: You can also use ones(m) and zeros(m) to produce square matrices.

- Matrix of random numbers: rand, randn
  - rand generates random numbers uniformly distributed in the range [0,1]
  - randn generates random numbers from the normal distribution with zero mean and variance one

```
>> eye(3)
ans =
Diagonal Matrix
   1    0    0
   0    1    0
   0    0    1
>> ones(3,2)
...
>> zeros(2,10)
...
```

```
>> rand(3,2)
ans =
   0.562728   0.057675
   0.697043   0.442021
   0.839662   0.310947
>> randn(3,2)
ans =
   1.12010  -0.96770
  -1.36156  -0.45994
   0.38406   2.33878
```

Arithmetic operations on matrices (1/2)

- Addition(+), subtraction(-), transpose(‘)

```
>> A+B‘
ans =
    2    4    6
    4    6    8
```

```
>> A'+B
ans =
    2    4
    4    6
    6    8
```

```
>> A+B
error: operator +: nonconformant
arguments (op1 is 2x3, op2 is
3x2)
```

- Mutiplication

```
>> C=A*B
ans =
   14   20
   20   29
```

```
>> D=B*A
ans =
    5    8   11
    8   13   18
   11   18   25
```

- Determinant

```
>> det(C)
ans =  6.0000
>> det(C')
ans =  6.0000
```

```
>> det(D)
ans =    1.7764e-15
```

Arithmetic operations on matrices (2/2)

- Element-wise product (.*) and division (./)

```
>> A.*B'
ans =
    1     4     9
    4     9    16
```

```
>> A./B'
ans =
    1     1     1
    1     1     1
```

- Power of a square matrix (^)

```
>> (A*A')^2
ans =
   596    860
   860   1241
```

```
>> A*A'
ans =
   14   20
   20   29
```

- Element-wise power (.^)

```
>> (A*A').^2
ans =
   196    400
   400    841
```

Norm of vectors and matrices

- Norm of a vector : norm(x,p)

$$\|\mathbf{x}\|_p = \sqrt[p]{\sum_{i=1}^{m} x_i^p} \qquad \mathbf{x} = [x_1, x_2, \cdots, x_m]^\top$$

```
>> x=[1,3,2];
>> norm(x)
ans =     3.7417
>> norm(x,2)
ans =     3.7417
>> norm(x,1)
ans =      6
>> norm(x,inf)
ans =      3
```

$$\|\mathbf{x}\|_2 = \|\mathbf{x}\| = \sqrt{\sum_{i=1}^{m} x_i^2}$$

$$\|\mathbf{x}\|_1 = \sum_{i=1}^{m} |x_i|$$

$$\|\mathbf{x}\|_\infty = \max(x_1, \cdots, x_m)$$

- Norm of a matrix : norm(X,p)
    - E.g., Frobenius norm*

$$\|\mathbf{X}\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}^2} = \sqrt{\mathrm{trace}(\mathbf{X}^\top \mathbf{X})}$$

```
>> X=randn(3,4);
>> norm(X,'fro')
ans =
     3,4349
```

```
>> sqrt(trace(X*X'))
ans =
     3.4349
```

*https://en.wikipedia.org/wiki/Matrix_norm

Inverse matrices

- The inverse A^{-1} of a square matrix A can be calculated by inv

```
>> A=randn(3,3)
A =
   0.087948   1.279500   0.060176
  -1.494407  -0.188317  -0.918068
  -1.063032   1.306333   0.734150
>> B=inv(A)
B =
   0.4055585  -0.3289932  -0.4446546
   0.7923708   0.0491297  -0.0035107
  -0.8226907  -0.5637950   0.7245167
>> B*A
ans =
   1.00000   0.00000  -0.00000
  -0.00000   1.00000   0.00000
   0.00000  -0.00000   1.00000
>> A*B
ans =
   1.00000   0.00000   0.00000
   0.00000   1.00000   0.00000
  -0.00000   0.00000   1.00000
```

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

Linear equations

- Use operator '¥' (Gaussian elimination) or inversion inv

```
>> A=[2,2,1;3,-1,3;2,-1,-3]
A =
    2    2    1
    3   -1    3
    2   -1   -3
>> b=[0;3;-1]
b =
    0
    3
   -1
>> A¥b
ans =
    0.19512
   -0.51220
    0.63415
>> inv(A)*b
ans =
    0.19512
   -0.51220
    0.63415
```

Remark: In general, inverse matrices should not be used for solving linear equations, particularly very large ones, from the perspective of computational efficiency and numerical accuracy

A simultaneous equation:

$$2x + 2y + z = 0$$
$$3x - y + 3z = 3$$
$$2x - y - 3z = -1$$

Its vector-matrix notation:

$$\begin{bmatrix} 2 & 2 & 1 \\ 3 & -1 & 3 \\ 2 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$$

The solution:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0.19512 \\ -0.51220 \\ 0.63415 \end{bmatrix}$$

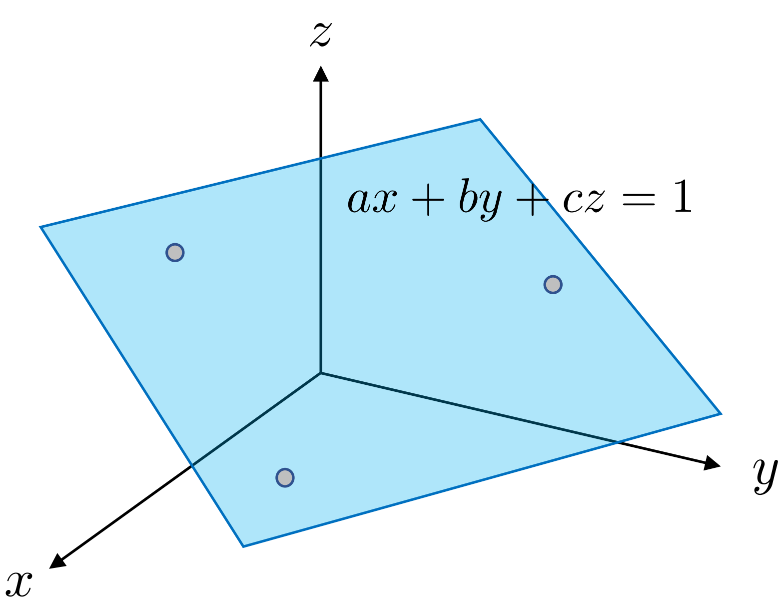
Gaussian elimination*

System of equations	Row operations	Augmented matrix
$\begin{aligned} 2x + y - z &= 8 \\ -3x - y + 2z &= -11 \\ -2x + y + 2z &= -3 \end{aligned}$		$\left[\begin{array}{ccc\|c} 2 & 1 & -1 & 8 \\ -3 & -1 & 2 & -11 \\ -2 & 1 & 2 & -3 \end{array}\right]$
$\begin{aligned} 2x + y - z &= 8 \\ \frac{1}{2}y + \frac{1}{2}z &= 1 \\ 2y + z &= 5 \end{aligned}$	$L_2 + \frac{3}{2}L_1 \to L_2$ $L_3 + L_1 \to L_3$	$\left[\begin{array}{ccc\|c} 2 & 1 & -1 & 8 \\ 0 & 1/2 & 1/2 & 1 \\ 0 & 2 & 1 & 5 \end{array}\right]$
$\begin{aligned} 2x + y - z &= 8 \\ \frac{1}{2}y + \frac{1}{2}z &= 1 \\ -z &= 1 \end{aligned}$	$L_3 + -4L_2 \to L_3$	$\left[\begin{array}{ccc\|c} 2 & 1 & -1 & 8 \\ 0 & 1/2 & 1/2 & 1 \\ 0 & 0 & -1 & 1 \end{array}\right]$
The matrix is now in echelon form (also called triangular form)		
$\begin{aligned} 2x + y &= 7 \\ \frac{1}{2}y &= \frac{3}{2} \\ -z &= 1 \end{aligned}$	$L_2 + \frac{1}{2}L_3 \to L_2$ $L_1 - L_3 \to L_1$	$\left[\begin{array}{ccc\|c} 2 & 1 & 0 & 7 \\ 0 & 1/2 & 0 & 3/2 \\ 0 & 0 & -1 & 1 \end{array}\right]$
$\begin{aligned} 2x + y &= 7 \\ y &= 3 \\ z &= -1 \end{aligned}$	$2L_2 \to L_2$ $-L_3 \to L_3$	$\left[\begin{array}{ccc\|c} 2 & 1 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$
$\begin{aligned} x &= 2 \\ y &= 3 \\ z &= -1 \end{aligned}$	$L_1 - L_2 \to L_1$ $\frac{1}{2}L_1 \to L_1$	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$

*https://en.wikipedia.org/wiki/Gaussian_elimination

Exercises 3.1

- Suppose we have three points in 3D space and their coordinates are $(x,y,z)=(0.2+r_{x1}, -0.1+r_{y1}, 1.0+r_{z1})$, $(3.0+r_{x2}, 0.1+r_{y2}, -1.0+r_{z2})$, and $(1.0+r_{x3}, -2.0+r_{y3}, -0.5+r_{z3})$, respectively. r is a random number between -0.1 and 0.1. Find a plane passing through these three points. Note that the equation of a plane that does not pass through the origin (0,0,0) is given by $ax + by + cz = 1$



A plane in 3D space passing through three points and not through the origin

Hint : Set up simultaneous linear equations and solve it to determine unknowns (a,b,c)