10. Matrices and linear algebra II
行列と線形代数II

- Eigenvectors and eigenvalues
  固有ベクトルと固有値

- Singular value decomposition
  特異値分解

- Rank of a matrix
  行列のランク

- Low-rank approximation of matrices
  行列の低ランク近似

Eigenvalues and eigenvectors
固有値と固有ベクトル

- `[V,D]=eig(A)`: calculates eigenvectors and eigenvalues of a square matrix
  `[V,D]=eig(A)`: 正方行列の固有ベクトルと固有値を計算します.
  - Eigenvectors are stored in ascending order in a diagonal matrix
    固有ベクトルは対角行列に昇順で格納されます.

$$\underbrace{\mathbf{A}\mathbf{v}_i}_{\text{eigenvector 固有ベクトル}} = \underbrace{d_i}_{\text{eigenvalue 固有値}}\mathbf{v}_i \iff \mathbf{AV} = \mathbf{VD}$$

$$\mathbf{V} = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n\end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$$

```
>> A=randn(3,3);
>> [V,D]=eig(A)
V =
   0.52988 + 0.00000i  -0.05375 - 0.34548i  -0.05375 + 0.34548i
   0.68932 + 0.00000i   0.84473 + 0.00000i   0.84473 - 0.00000i
   0.49404 + 0.00000i   0.12431 + 0.38565i   0.12431 - 0.38565i
D =
Diagonal Matrix
   0.04533 + 0.00000i                    0                    0
                    0   2.09047 + 1.25277i                    0
                    0                    0   2.09047 - 1.25277i
>> A*V-V*D
ans =
   1.6306e-16 + 0.0000e+00i  -1.1102e-15 - 2.2204e-15i  -1.1102e-15 + 2.2204e-15i
  -1.5959e-16 + 0.0000e+00i   0.0000e+00 + 4.4409e-16i   0.0000e+00 - 4.4409e-16i
  -2.6368e-16 + 0.0000e+00i  -1.1102e-16 + 3.3307e-16i  -1.1102e-16 - 3.3307e-16i
```

Eigenvectors/values of symmetric matrices 対称行列の固有ベクトルと固有値

- Symmetric matrices always have *real* eigenvectors/values
  対称行列は常に実固有ベクトル/固有値を持つ
  - Nonsymmetric matrices have complex eigenvectors/values in general as in the last slide
    非対称行列は一般に複素固有ベクトル/固有値を持つ
  - Many matrices we encounter in engineering will be symmetric
    工学の分野における多くの行列は対称行列である
  - Remark: their eigenvectors are always orthogonal, i.e., $\mathbf{V}^\top\mathbf{V} = \mathbf{V}\mathbf{V}^\top = \mathbf{I}$
    備考：対称行列の固有ベクトルは常に直交する、すなわち$\mathbf{V}^\top\mathbf{V} = \mathbf{V}\mathbf{V}^\top = \mathbf{I}$
  - The symmetric matrix is 'diagonalized' by V as $\mathbf{V}^\top\mathbf{A}\mathbf{V} = \mathbf{D}$
    対称行列は, $\mathbf{V}^\top\mathbf{A}\mathbf{V} = \mathbf{D}$ のようにVによって「対角化」されます。

```
>> X = randn(3,3);
>> A=X'*X;
>> [V,D]=eig(A)
V =
    0.960179    0.267639    0.080159
   -0.226697    0.914040   -0.336363
   -0.163292    0.304796    0.938315

D =
Diagonal Matrix
    0.015584           0           0
           0    1.752624           0
           0           0    6.254892
```

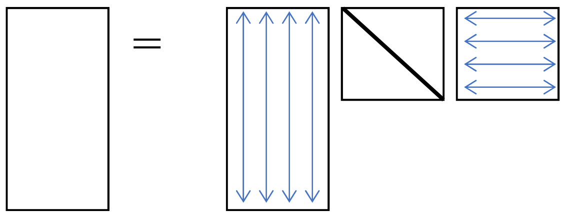
```
>> A*V-V*D
ans =
   1.2143e-17  -2.7756e-16   3.3307e-16
   9.1507e-17   0.0000e+00  -4.4409e-16
  -1.5179e-16   0.0000e+00   0.0000e+00

>> V'*A*V
ans =
   1.5584e-02   4.2718e-17  -1.7391e-16
  -1.3878e-17   1.7526e+00   2.2204e-16
  -2.2204e-16   4.4409e-16   6.2549e+00
```

Singular value decomposition of matrices (1/2)
行列の特異値分解 (1/2)

- Any $m \times n$ real matrix can be decomposed into a product of orthogonal matrices U and V and a diagonal matrix W as follows:
  以下のように，任意の$m \times n$実数行列は直交行列Ｕおよび直交行列Ｖと対角行列Ｗとの積に分解することができる.

$$\mathbf{X} = \mathbf{U}\mathbf{W}\mathbf{V}^\top$$



singular values
Xの特異値

$$\mathbf{W} = \begin{bmatrix} w_1 & & & \\ & w_2 & & \\ & & \ddots & \\ & & & w_n \end{bmatrix}$$

$$\mathbf{U}^\top\mathbf{U} = \mathbf{I} \qquad \mathbf{V}^\top\mathbf{V} = \mathbf{V}\mathbf{V}^\top = \mathbf{I}$$

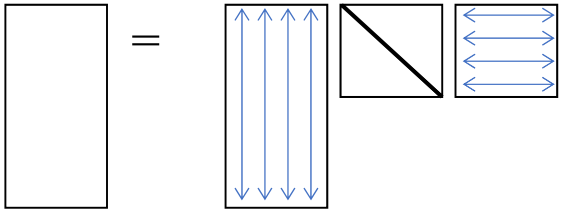
- Remark: The decomposition is *unique* when we fix the order of the singular values (say, in descending order)
  注意：特異値の順序を（例えば，降順に）固定すると，特異値分解は一意になる.

Singular value decomposition of matrices (2/2) 行列の特異値分解 (2/2)

- `svd`: calculates singular value decomposition
  `svd`: 特異値分解を計算する関数
  - Singular value decomposition is often abbreviated as SVD
    特異値分解はSVDと略される

```
>> X=randn(5,3);
>> [U,W,V]=svd(X,0);
>> W
W =
Diagonal Matrix
   3.07321         0         0
         0   1.73673         0
         0         0   0.82822
```

$$\mathbf{X} = \mathbf{U}\mathbf{W}\mathbf{V}^\top$$



```
>> norm(U*W*V'-X)
ans =    1.5822e-15
>> norm(U'*U-eye(3))
ans =    6.7963e-16
>> norm(V'*V-eye(3))
ans =    2.3629e-16
```

Relation between SVD and eigenproblem
SVDと固有値の関係

- Column vectors of V of SVD of X coincides with eigenvecoters of A=X’X
  XのSVDのVの列ベクトルは，A = X‘X の固有ベクトルと一致する．

$$\begin{aligned}\mathbf{A} &= \mathbf{X}^\top\mathbf{X}\\ &= (\mathbf{U}\mathbf{W}\mathbf{V}^\top)^\top(\mathbf{U}\mathbf{W}\mathbf{V}^\top)\\ &= \mathbf{V}\mathbf{W}^\top\mathbf{U}^\top\mathbf{U}\mathbf{W}\mathbf{V}^\top\\ &= \mathbf{V}\mathbf{W}^2\mathbf{V}^\top\end{aligned}$$

- Singular values of X are equal to the square roots of eigenvalues of A=X’X
  Xの特異値は，A = X’Xの固有値の平方根に等しい

```
>> sqrt(D)
ans =
Diagonal Matrix
    1.9851         0         0
         0    3.9417         0
         0         0    4.7675
```

```
>> X=randn(10,3);
>> [V1,D]=eig(X'*X);
>> [U,W,V2]=svd(X,0)
>> V1
V1 =
  -0.69324  -0.61974  -0.36789
   0.14611   0.37901  -0.91379
   0.70574  -0.68722  -0.17219

>> V2
V2 =
   0.36789   0.61974   0.69324
   0.91379  -0.37901  -0.14611
   0.17219   0.68722  -0.70574
```

```
>> W
W =
Diagonal Matrix
   4.7675        0        0
        0   3.9417        0
        0        0   1.9851
        0        0        0
        0        0        0
        0        0        0
        0        0        0
        0        0        0
        0        0        0
        0        0        0
```

Properties of SVD
SVDの特性

- Pseudo inverse of X can be written using its SVD as
  Xの擬似逆行列は，SVDを使用して次のように書くことができる.

$$\mathbf{X} = \mathbf{U}\mathbf{W}\mathbf{V}^\top$$

$$\downarrow$$

$$\mathbf{X}^\dagger = \mathbf{V}\mathbf{W}^{-1}\mathbf{U}^\top$$

- The number of non-zero singular values of X is called the rank of X
  XのゼロでないA特異値の数はXのランクと呼ばれます

```
>> X=randn(5,3);
>> pinv(X)
ans =
   0.037163  -0.070115  -0.329386   0.373801  -0.323277
  -0.116157  -0.215984  -0.339899  -0.014697  -0.207555
  -0.066574  -0.156025   0.513828   0.023533  -0.501137

>> [U,W,V]=svd(X,0);
>> V*inv(W)*U'
ans =
   0.037163  -0.070115  -0.329386   0.373801  -0.323277
  -0.116157  -0.215984  -0.339899  -0.014697  -0.207555
  -0.066574  -0.156025   0.513828   0.023533  -0.501137
```

```
>> X=randn(5,2)*randn(2,4)
X =
  -0.065735  -0.053739   1.626185   1.734253
  -0.022809  -0.020869   0.637444   0.672717
   0.151451   0.140834  -4.307108  -4.539055
   0.563733   0.153728  -3.832887  -5.067102
  -0.246376  -0.082560   2.181361   2.705379
>> rank(X)
ans =  2
>> svd(X)
ans =
   9.9100e+00
   6.0159e-01
   2.4902e-16
   1.6489e-17
```

Approximation of matrices by SVD
SVDによる行列の近似

- Consider the following problem: given a matrix A, we wish to obtain a matrix of a fixed rank *r* that approximates A as accurately as possible
  問題：行列Aが与えられたとき，Aをできるだけ正確に近似する固定ランクrの行列を求めよ

- It can be formulated as a *constrained minimization* problem:
  これは，以下のように制約付き最小化問題として定式化できる

$$\min_{\hat{\mathbf{A}}} \|\mathbf{A} - \hat{\mathbf{A}}\|_F \quad \text{subject to} \quad \mathrm{rank}(\mathbf{A}) = r$$

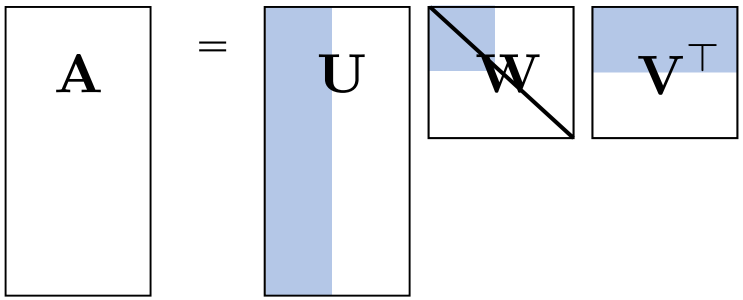
- Its solution is simply given by SVD of A in the following way:
  解は，以下のようにAのSVDによって求められます

$$\mathbf{A} = \mathbf{U}\mathbf{W}\mathbf{V}^\top$$

$$\mathbf{U} = [\mathbf{u}_1, \ldots, \mathbf{u}_r, \ldots, \mathbf{u}_n]$$

$$\mathbf{W} = \mathrm{diag}[w_1, \ldots, w_r, \ldots, w_n]$$

$$\mathbf{V} = [\mathbf{v}_1, \ldots, \mathbf{v}_r, \ldots, \mathbf{v}_n]$$



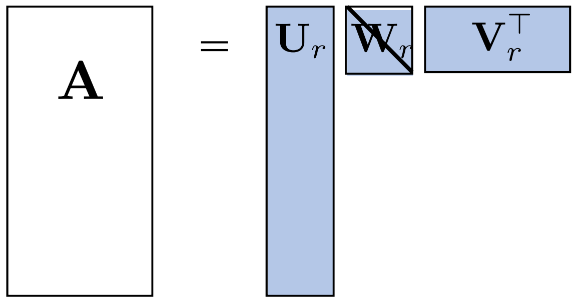
Simply remove $(r{+}1)^{\text{th}}$ to n^{th} column vectors
単に（r + 1）番目からn番目までの列ベクトルを削除する

$$\hat{\mathbf{A}} = \mathbf{U}_r\mathbf{W}_r\mathbf{V}_r^\top$$

$$\mathbf{U}_r = [\mathbf{u}_1, \ldots, \mathbf{u}_r]$$

$$\mathbf{W}_r = \mathrm{diag}[w_1, \ldots, w_r]$$

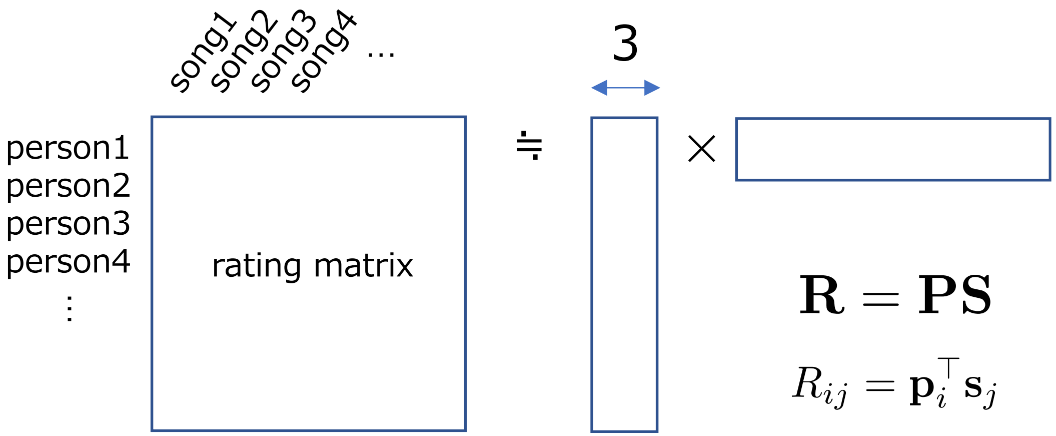
$$\mathbf{V}_r = [\mathbf{v}_1, \ldots, \mathbf{v}_r]$$



Exercise 10.1

- ある人がどのように曲を評価するかを予測したい

- 音楽の好き/嫌いについては同じような好みを持つ人々がいます
  - とは言っても，まったく同じ好みを持つ人物は二人といない
  - この種の問題は協調フィルタリングとして知られている
- 評価行列Rをrank = 3の行列PおよびSで近似する



$$\mathbf{R} = \mathbf{P}\mathbf{S}$$

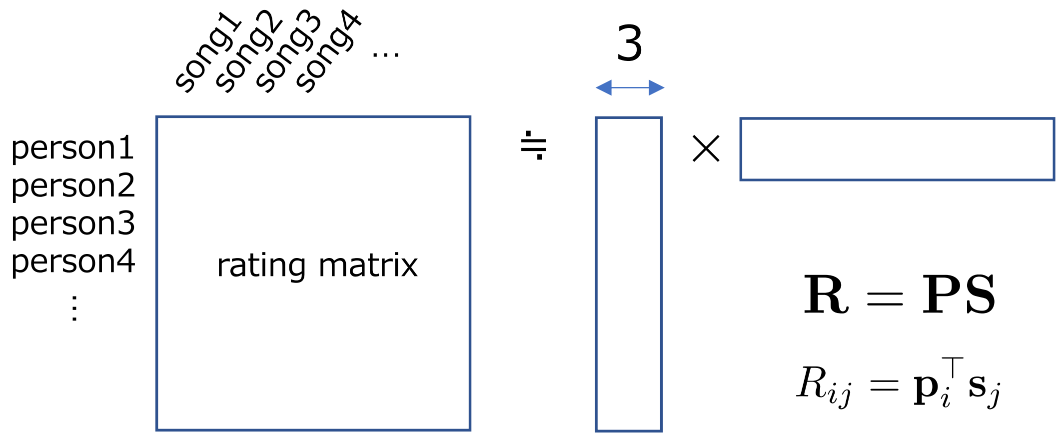
$$R_{ij} = \mathbf{p}_i^\top \mathbf{s}_j$$

$$\mathbf{P} = \begin{bmatrix} \mathbf{p}_1^\top \\ \mathbf{p}_2^\top \\ \vdots \\ \mathbf{p}_{15}^\top \end{bmatrix} \qquad \mathbf{S} = [\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_{20}]$$

Exercise 10.1

- We wish to predict how a person rates songs

- Some people have similar tastes about like/dislike of music
  - That said, there will be no two persons having exactly the same taste
  - This kind of problems is known as *collaborative filtering*
- We approximate the rating matrix by a matrix of rank=3



$$\mathbf{P} = \begin{bmatrix} \mathbf{p}_1^\top \\ \mathbf{p}_2^\top \\ \vdots \\ \mathbf{p}_{15}^\top \end{bmatrix} \qquad \mathbf{S} = [\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_{20}]$$

$$\mathbf{R} = \mathbf{P}\mathbf{S}$$

$$R_{ij} = \mathbf{p}_i^\top \mathbf{s}_j$$

Exercise 10.1

- 15人による20曲の評価を利用する
  - コースのページにある `rating.txt`というファイルをダウンロードし，Octaveで読み込む

```
>> load('rating.txt')
```

  - 評価は[1,5]の範囲の整数で表される
  - R（2,4）＝ 3は、person2がsong4に対して評価=3を与えたことを意味します
- 新しい人（すなわち１６番目の人）が３曲の評価を以下のように与えたと仮定する
  - song1=4, song3=2, song7=3, すなわち, $R_{16,1}=4,\ R_{16,3}=2,\ R_{16,7}=3$
- この人による他の曲の評価を推定する
  - まず，Ｒのランク３近似を見つける．すなわち，15x3 P行列 および 3x20 S行列を得る
  - 次に，行列Sを用いて次の式を満たすp_{16}を求める

$$
\begin{aligned}
R_{16,1} &= \mathbf{p}_{16}^\top \mathbf{s}_1 \\
R_{16,3} &= \mathbf{p}_{16}^\top \mathbf{s}_3 \\
R_{16,7} &= \mathbf{p}_{16}^\top \mathbf{s}_7
\end{aligned}
$$

  - 最後に，右式によって評価の予測を計算する $R_{16,j} = \mathbf{p}_{16}^\top \mathbf{s}_j$
  - 実際の評価は以下である

4 3 2 2 3 3 3 2 3 1 2 3 2 2 3 4 3 3 3 3

Exercise 10.1

- Ratings of 20 songs by 15 persons are available
  - Download `rating.txt` from the course page and read into `R` by

```
>> load('rating.txt')
```

  - Rating is represented by an integer in the range of [1,5]
  - R(2,4)=3 means person2 gave rating=3 for song4
- Suppose a new (i.e., 16th) person gives ratings for three songs
  - song1=4, song3=2, song7=3, i.e., $R_{16,1} = 4,\ R_{16,3} = 2,\ R_{16,7} = 3$
- Estimate ratings by this person for other songs
  - First, find a rank-3 approximation of R, i.e., obtain 15x3 P and 3x20 S
  - Second, find p_{16} that satisfies the following equations using S:

$$R_{16,1} = \mathbf{p}_{16}^\top \mathbf{s}_1$$
$$R_{16,3} = \mathbf{p}_{16}^\top \mathbf{s}_3$$
$$R_{16,7} = \mathbf{p}_{16}^\top \mathbf{s}_7$$

  - Finally, calculate prediction of ratings by $R_{16,j} = \mathbf{p}_{16}^\top \mathbf{s}_j$
  - True ratings are:

4 3 2 2 3 3 3 2 3 1 2 3 2 2 3 4 3 3 3 3