6. Numerical integration and ordinary differential equation

- Numerical integration (definite integral)
- Double integral
- Initial value problem of ODEs

Numerical integration

- The value of a definite integral can be calculated using quad
- E.g., To calculate the following definite integral:

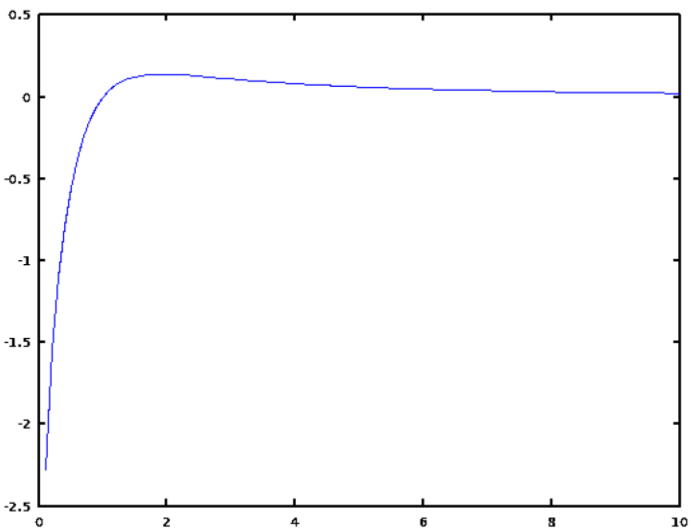
$$\int_0^{10} \frac{\log x}{1+x^2} dx$$

```
>> quad(@(x)(log(x)/(1+x^2)), 0, 10)
ans = -0.32938
```

- You can plot the original function by

```
>> x=0:0.1:10;
>> plot(x, log(x)./(1+x.^2)))
```

Remark: Recall element-wise operations of matrices/vectors have a preceding period, e.g., './' and '.^'



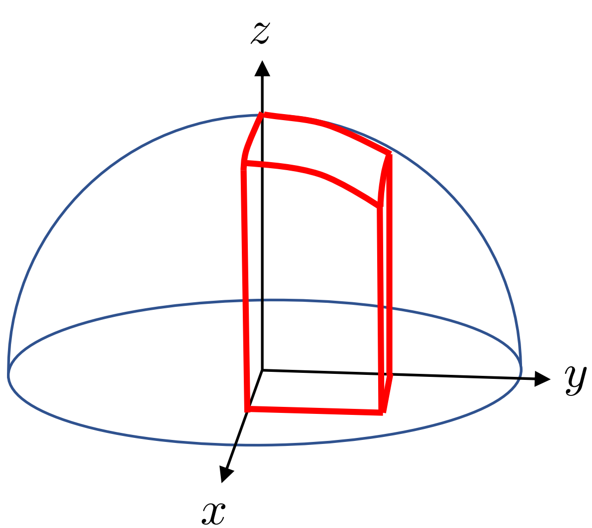
Double integral

- The value of double integral can be calculated using dblquad
- E.g., To calculate the volume of a part of the hemisphere of a unit sphere

$$x^2 + y^2 + z^2 = 1$$

$$z = \sqrt{1 - x^2 - y^2}$$

```
>> dblquad(@(x,y)(sqrt(1-x.^2-y.^2)),0,0.5,0,0.5)
ans =  0.22774
```



Initial value problem of ODEs

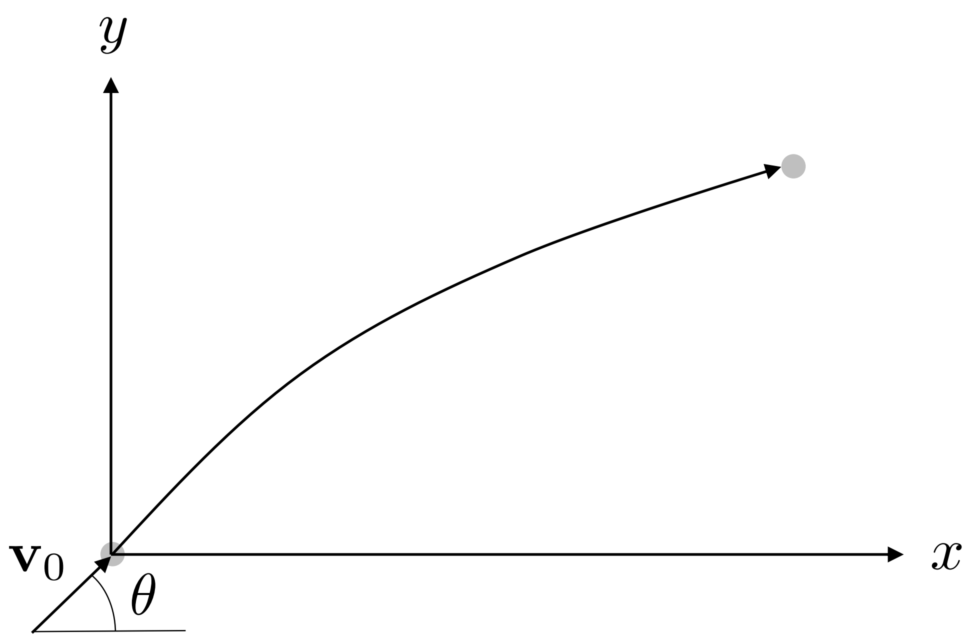
- Four steps to solve an initial value problem of an ODE
  1. Derive differential equations describing the target system
  2. If they are 2nd and higher order ODEs, convert them into a system of 1st order ODEs by incorporating new variables
  3. Create a function (a script file) that calculates the derivatives of the variables from their values and time
  4. Calculate how each variable changes with time using function ode45 by providing it with initial values of the variables and a time interval to consider.

Example

- Suppose that a metal ball with mass m [kg] is thrown into space with elevation angle θ [rad] and initial velocity v_0 [m/s]
- The equation of motion is represented with coordinates (x,y) as

$$\frac{d^2x}{dt^2} = 0$$ (Const. velocity)

$$\frac{d^2y}{dt^2} = -g$$ (Standard acceleration due to gravity)



How to solve the example problem (1/2)

- Let (v_x,v_y) be the velocities of the ball in the *x and y* axis, respectively
- Convert the equations in the last page into the 1st order diffenretial eq. wrt. *x, y,* v_x, and v_y

$$\frac{dx}{dt} = v_x \qquad \frac{dy}{dt} = v_y \qquad \frac{dv_x}{dt} = 0 \qquad \frac{dv_y}{dt} = -g$$

- Create a function that calculates these derivatives
  - Let *p* be a 4-vector storing *x, y,* v_x, v_y at time *t*

$$\mathbf{p} = (x, y, v_x, v_y)$$

  - Write a function that calculates the derivative *d***p***/dt* from *t* and **p**

```
function dp = deriv_fun(t, p)
g = 9.81;
dp = [p(3), p(4), 0, -g];
```

$$\frac{d\mathbf{p}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dv_x}{dt}, \frac{dv_y}{dt}\right)$$

How to solve the example problem (2/2)

- Call function `ode45` with a time interval and initial values

Only in older Octave versions

User-defined func. of $d\mathbf{p}/dt$

Time interval

Initial values of x, y, v_x, v_y at $t=0$

$\mathbf{v}_0 = (v_0 \cos\theta, v_0 \sin\theta)$

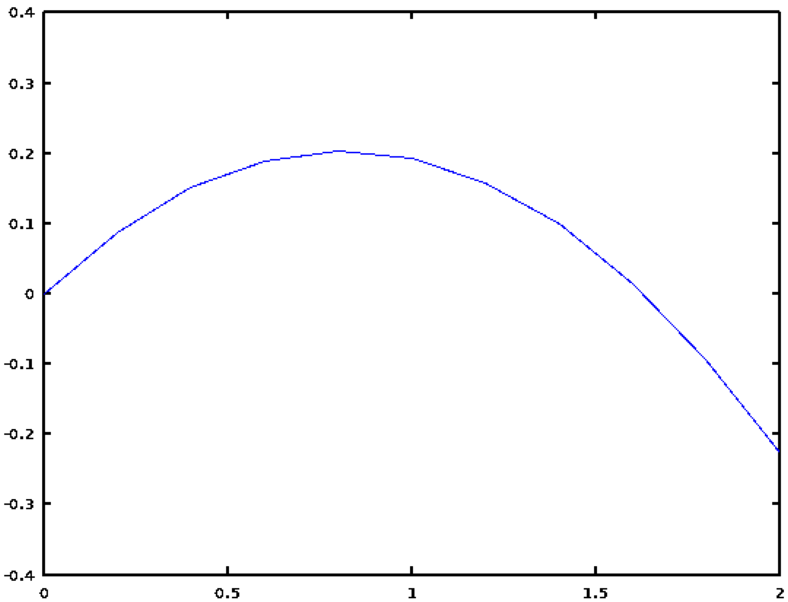
```
>> pkg load odepkg
>> [T, result] = ode45(@deriv_fun, [0,0.5], [0,0,4.0,2.0])
```

Results:

```
warning: Option "RelTol" not set, new value 0.000001 is used
warning: called from    ode45 at line 113 column 5
warning: Option "AbsTol" not set, new value 0.000001 is used
warning: Option "InitialStep" not set, new value 0.050000 is used
warning: Option "MaxStep" not set, new value 0.050000 is used
T =
   0.00000
   0.05000
   0.10000
   0.15000
   0.20000
   0.25000
   0.30000
   0.35000
   0.40000
   0.45000
   0.50000
   0.50000
result =
   0.00000   0.00000   4.00000   2.00000
   0.20000   0.08774   4.00000   1.50950
   0.40000   0.15095   4.00000   1.01900
   0.60000   0.18964   4.00000   0.52850
   0.80000   0.20380   4.00000   0.03800
   1.00000   0.19344   4.00000  -0.45250
   1.20000   0.15855   4.00000  -0.94300
   1.40000   0.09914   4.00000  -1.43350
   1.60000   0.01520   4.00000  -1.92400
   1.80000  -0.09326   4.00000  -2.41450
   2.00000  -0.22625   4.00000  -2.90500
   2.00000  -0.22625   4.00000  -2.90500
```

Plot of a trajectory of the metal ball

```
>> plot(result(:,1), result(:,2))
```

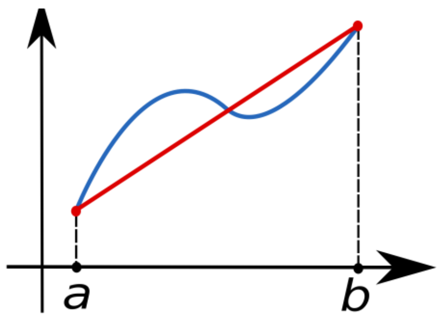


Quadrature rules and Runge-Kutta method*

- Definite integral is numerically computed by several approximation methods, e.g., the trapezoidal rule or Simpson rule

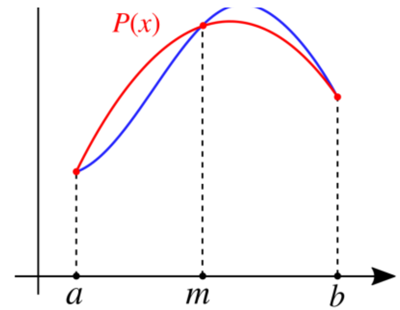
The trapezoidal rule

$$\int_a^b f(x)\,\mathrm{d}x \approx (b-a)\frac{f(a)+f(b)}{2}$$



Simpson rule

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$

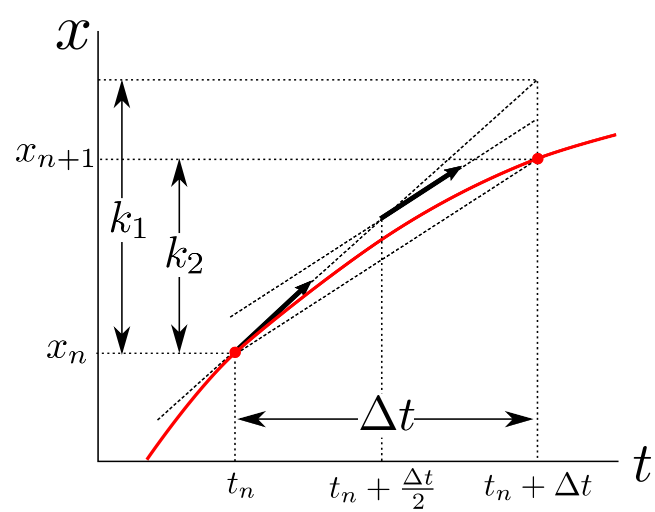


- The core of numerical solutoins to ODEs is numerical integration
    - 2nd order Runge-Kutta method

$$k_1 = \Delta t\, f(t_n, x_n)$$

$$k_2 = \Delta t\, f(t_n + \frac{1}{2}\Delta t, x_n + \frac{1}{2}k_1)$$

$$x_{n+1} = x_n + k_2(+O(\Delta t^3))$$



Exercise 6.1

Consider a mass *m, to which* a spring with spring constant *k* and a damper with damping constant *c are attached* as shown in the diagram. Assume that the mass can move only in the *x*. The equation of motion is given by

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

When setting *c* to ((your birth month) modulo 3)+1) and *k* to ((your birth day) modulo 7)+1), plot $x(t)$ with $m=1$, $x(0)=1$ and $dx/dt(0)=0$.

E.g., If your birth month and date is 13th August, then c=3 and k=7