

3. Matrices and linear algebra I

- Accessing elements
- Basic operations
- Norms
- Inverse matrix
- Linear equation

Accessing elements

- As you have learned, ' ; ' indicates the end of a row; matrices of any size can be created in this way

```
>> A=[1,2,3;2,3,4]  
A =  
1 2 3  
2 3 4
```

```
>> B=[1,2;2,3;3,4]  
B =  
1 2  
2 3  
3 4
```

- Specify row and column indices to access an element
- A whole row or a whole column can be represented using ':'

```
>> A(2,3)  
ans = 4  
>> A(1,2)  
ans = 2
```

```
>> B(3,:)  
ans =  
3 4
```

```
>> B(:,1)  
ans =  
1  
2  
3
```

Quick creation of several matrices by functions

- Identity matrix: eye (m)
- Matrix of all 1's: ones(m,n)
- Matrix of all 0's: zeros(m,n)
- Matrix of random numbers: rand, randn
 - rand generates random numbers uniformly distributed in the range [0,1]
 - randn generates random numbers from the normal distribution with zero mean and variance one

```
>> eye(3)
ans =
Diagonal Matrix
    1     0     0
    0     1     0
    0     0     1
>> ones(3,2)
...
>> zeros(2,10)
...
```

```
>> rand(3,2)
ans =
    0.562728    0.057675
    0.697043    0.442021
    0.839662    0.310947
>> randn(3,2)
ans =
    1.12010   -0.96770
   -1.36156   -0.45994
    0.38406    2.33878
```

Arithmetc operations on matrices (1/2)

- Addition(+), subtraction(-), transpose(')

```
>> A+B'  
ans =  
2 4 6  
4 6 8
```

```
>> A'+B  
ans =  
2 4  
4 6  
6 8
```

```
>> A+B  
error: operator +: nonconformant  
arguments (op1 is 2x3, op2 is  
3x2)
```

- Mutiplication

```
>> C=A*B  
ans =  
14 20  
20 29
```

```
>> D=B*A  
ans =  
5 8 11  
8 13 18  
11 18 25
```

- Determinant

```
>> det(C)  
ans = 6.0000  
>> det(C')  
ans = 6.0000
```

```
>> det(D)  
ans = 1.7764e-15
```

Arithmetic operations on matrices (2/2)

- Element-wise product (.*) and division (./)

```
>> A.*B'  
ans =  
1 4 9  
4 9 16
```

```
>> A./B'  
ans =  
1 1 1  
1 1 1
```

- Power of a square matrix (^)

```
>> (A*A')^2  
ans =  
596 860  
860 1241
```

```
>> A*A'  
ans =  
14 20  
20 29
```

- Element-wise power (.^)

```
>> (A*A').^2  
ans =  
196 400  
400 841
```

Norm of vectors and matrices

- Norm of a vector : `norm(x,p)`

```
>> x=[1,3,2];
>> norm(x)
ans =      3.7417
>> norm(x,2)
ans =      3.7417
>> norm(x,1)
ans =       6
>> norm(x,inf)
ans =       3
```

$$\|\mathbf{x}\|_p = \sqrt[p]{\sum_{i=1}^m x_i^p} \quad \mathbf{x} = [x_1, x_2, \dots, x_m]^\top$$

$$\left. \begin{aligned} \|\mathbf{x}\|_2 &= \|\mathbf{x}\| = \sqrt{\sum_{i=1}^m x_i^2} \\ \|\mathbf{x}\|_1 &= \sum_{i=1}^m |x_i| \\ \|\mathbf{x}\|_\infty &= \max(x_1, \dots, x_m) \end{aligned} \right\}$$

- Norm of a matrix : `norm(X,p)`

- E.g., Frobenius norm

```
>> X=randn(3,4);
>> norm(X,'fro')
ans =
3.4349
```

$$\|\mathbf{X}\|_F = \sqrt{\sum_{i=1}^m \sum_{j=1}^n x_{ij}^2} = \sqrt{\text{trace}(\mathbf{X}^\top \mathbf{X})}$$

```
>> sqrt(trace(X*X'))
ans =
3.4349
```

Inverse matrices

- The inverse A^{-1} of a square matrix A can be calculated by inv

```
>> A=randn(3,3)
A =
    0.087948    1.279500    0.060176
   -1.494407   -0.188317   -0.918068
   -1.063032    1.306333    0.734150
>> B=inv(A)
B =
    0.4055585   -0.3289932   -0.4446546
    0.7923708     0.0491297   -0.0035107
   -0.8226907   -0.5637950    0.7245167
>> B*A
ans =
    1.00000    0.00000   -0.00000
   -0.00000    1.00000    0.00000
    0.00000   -0.00000    1.00000
>> A*B
ans =
    1.00000    0.00000    0.00000
    0.00000    1.00000    0.00000
   -0.00000   0.00000    1.00000
```

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

Linear equations

- Use operator '¥' (Gaussian elimination) or inversion inv

```
>> A=[2,2,1;3,-1,3;2,-1,-3]
A =
    2     2     1
    3    -1     3
    2    -1    -3
>> b=[0;3;-1]
b =
    0
    3
   -1
>> A¥b
ans =
    0.19512
   -0.51220
    0.63415
>> inv(A)*b
ans =
    0.19512
   -0.51220
    0.63415
```

Remark: In general, inverse matrices should not be used for solving linear equations, particularly very large ones, from the perspective of computational efficiency and numerical accuracy

A simultaneous equation:

$$2x + 2y + z = 0$$

$$3x - y + 3z = 3$$

$$2x - y - 3z = -1$$

Its vector-matrix notation:

$$\begin{bmatrix} 2 & 2 & 1 \\ 3 & -1 & 3 \\ 2 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$$

The solution:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0.19512 \\ -0.51220 \\ 0.63415 \end{bmatrix}$$

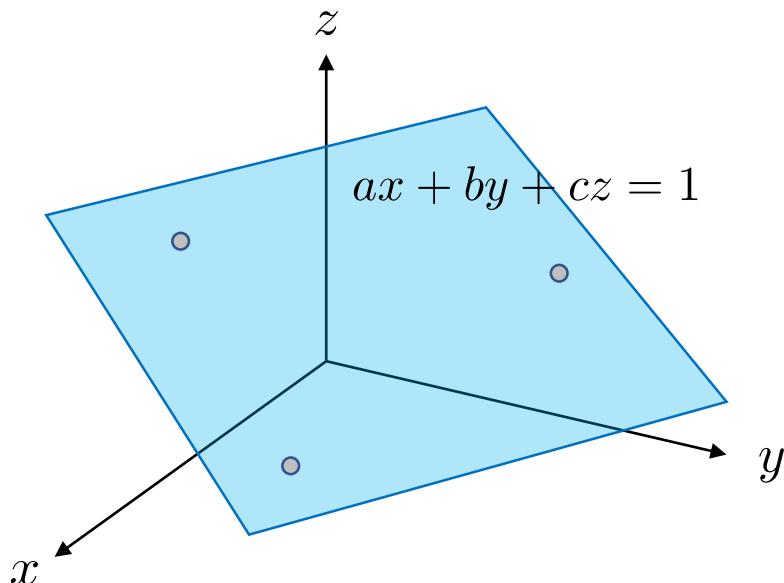
Gaussian elimination*

System of equations	Row operations	Augmented matrix
$\begin{aligned} 2x + y - z &= 8 \\ -3x - y + 2z &= -11 \\ -2x + y + 2z &= -3 \end{aligned}$		$\left[\begin{array}{ccc c} 2 & 1 & -1 & 8 \\ -3 & -1 & 2 & -11 \\ -2 & 1 & 2 & -3 \end{array} \right]$
$\begin{aligned} 2x + y - z &= 8 \\ \frac{1}{2}y + \frac{1}{2}z &= 1 \\ 2y + z &= 5 \end{aligned}$	$L_2 + \frac{3}{2}L_1 \rightarrow L_2$ $L_3 + L_1 \rightarrow L_3$	$\left[\begin{array}{ccc c} 2 & 1 & -1 & 8 \\ 0 & 1/2 & 1/2 & 1 \\ 0 & 2 & 1 & 5 \end{array} \right]$
$\begin{aligned} 2x + y - z &= 8 \\ \frac{1}{2}y + \frac{1}{2}z &= 1 \\ -z &= 1 \end{aligned}$	$L_3 + -4L_2 \rightarrow L_3$	$\left[\begin{array}{ccc c} 2 & 1 & -1 & 8 \\ 0 & 1/2 & 1/2 & 1 \\ 0 & 0 & -1 & 1 \end{array} \right]$
The matrix is now in echelon form (also called triangular form)		
$\begin{aligned} 2x + y &= 7 \\ \frac{1}{2}y &= \frac{3}{2} \\ -z &= 1 \end{aligned}$	$L_2 + \frac{1}{2}L_3 \rightarrow L_2$ $L_1 - L_3 \rightarrow L_1$	$\left[\begin{array}{ccc c} 2 & 1 & 0 & 7 \\ 0 & 1/2 & 0 & 3/2 \\ 0 & 0 & -1 & 1 \end{array} \right]$
$\begin{aligned} 2x + y &= 7 \\ y &= 3 \\ z &= -1 \end{aligned}$	$2L_2 \rightarrow L_2$ $-L_3 \rightarrow L_3$	$\left[\begin{array}{ccc c} 2 & 1 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array} \right]$
$\begin{aligned} x &= 2 \\ y &= 3 \\ z &= -1 \end{aligned}$	$L_1 - L_2 \rightarrow L_1$ $\frac{1}{2}L_1 \rightarrow L_1$	$\left[\begin{array}{ccc c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array} \right]$

Exercises 3.1

- Suppose we have three points in 3D space and their coordinates are $(x,y,z)=(0.2, -0.1, 1.0)$, $(3.0, 0.1, -1.0)$, and $(1.0, -2.0, -0.5)$, respectively. Find a plane passing through these three points. Note that the equation of a plane that does not pass through the origin $(0,0,0)$ is given by

$$ax + by + cz = 1$$



A plane in 3D space passing through three points and not through the origin

Hint : Set up simultaneous linear equations and solve it to determine unknowns (a,b,c)