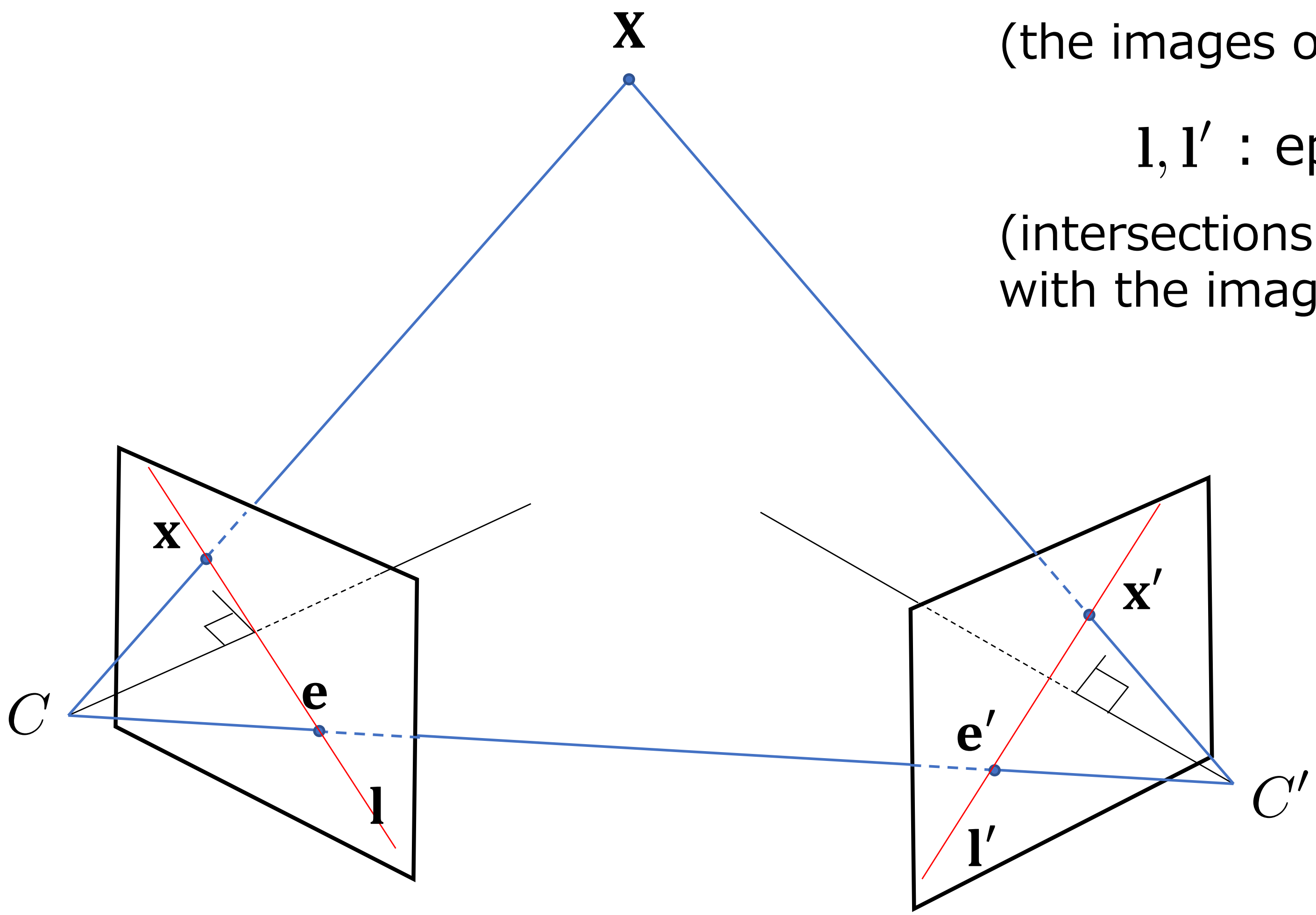
Two-view geometry

- epipolar points/lines
- fundamental matrix
- essential matrix
- estimation of relative camera pose

Two-view geometry

- Also known as *epipolar geometry*



$\mathbf{e}, \mathbf{e}'$: epipoles

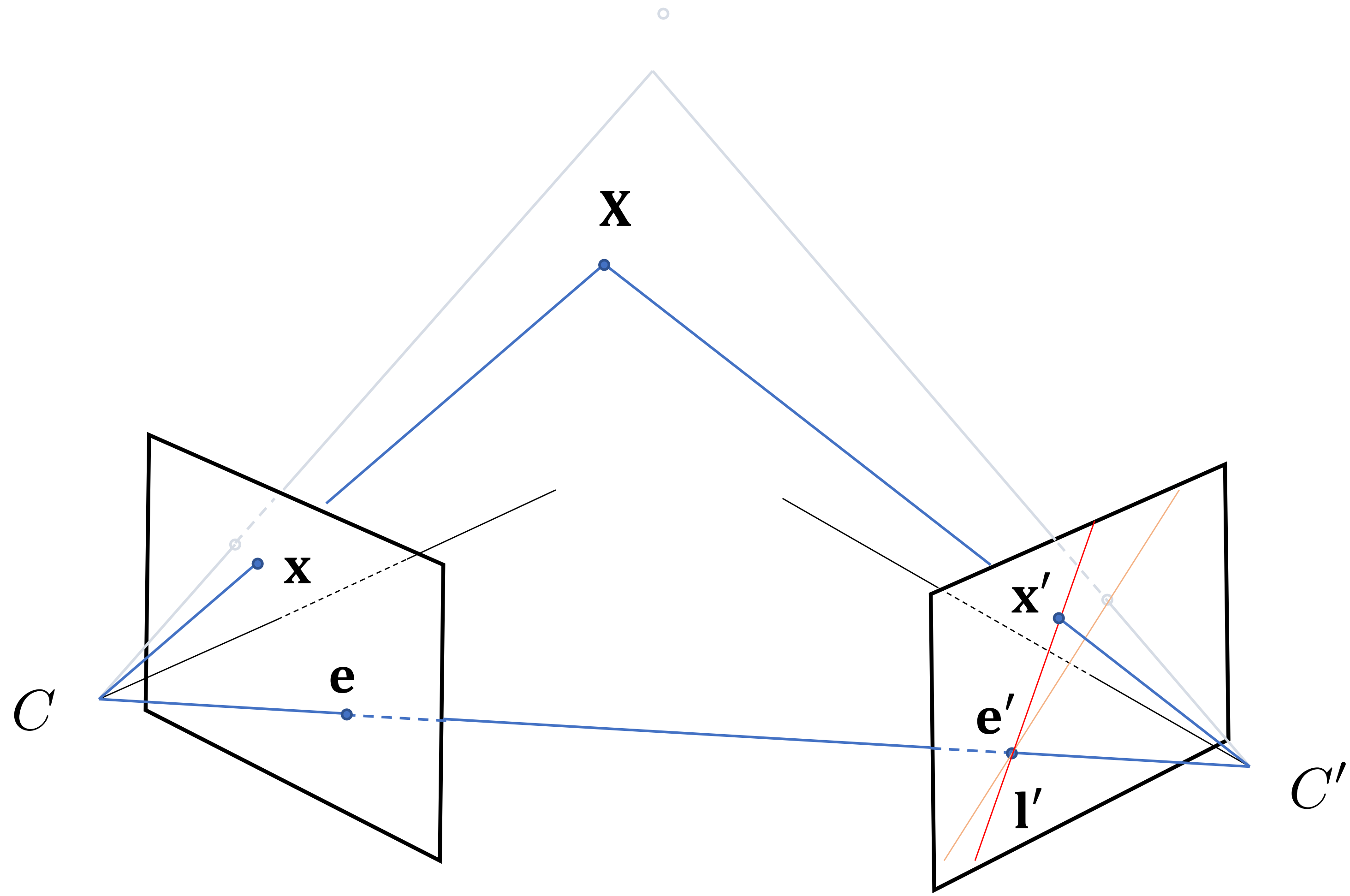
(the images of camera centers)

$\mathbf{l}, \mathbf{l}'$: epipolar lines

(intersections of the plane XCC′ with the image planes)

Two-view geometry

- Specifying $\mathbf{x}$, you have $\mathbf{l}'$



Two-view geometry

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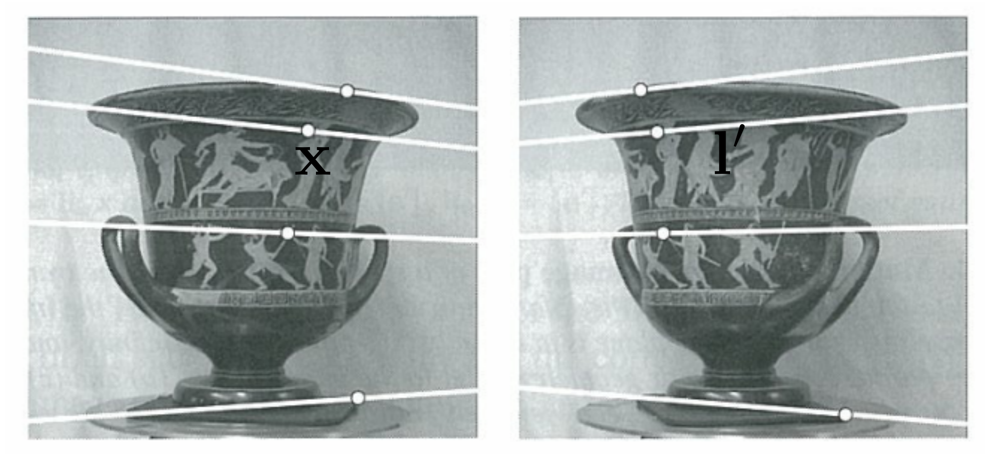
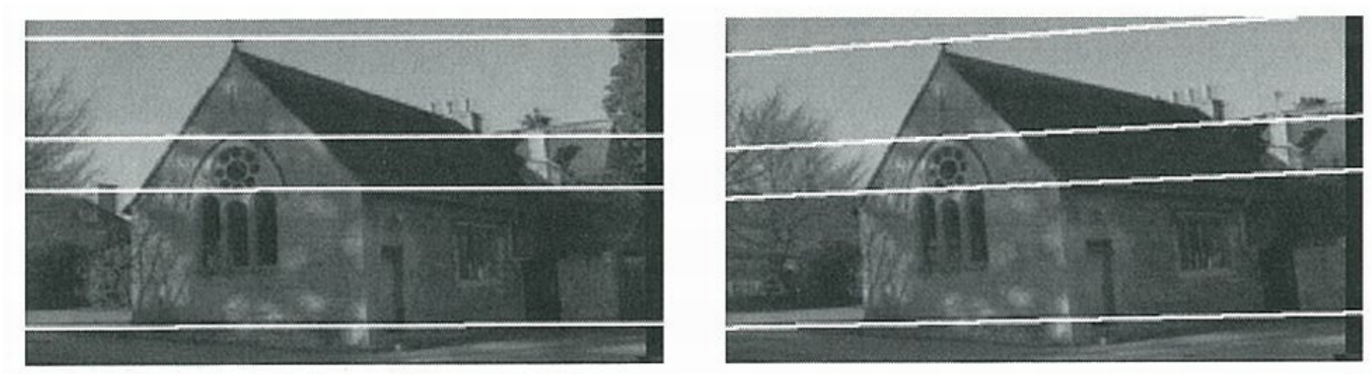


A case when the epipole is at infinity



[Hartley-Zisserman03]

Fundamental matrix

- The epipolar line $\mathbf{l}'$ specified by $\mathbf{x}$ is given as

$$\mathbf{l}' = \mathtt{F}\mathbf{x}$$

- $\mathtt{F}$ is a constant 3x3 matrix defined for a pair of views, which is called the fundamental matrix

Suppose a plane π that does not pass C or C'

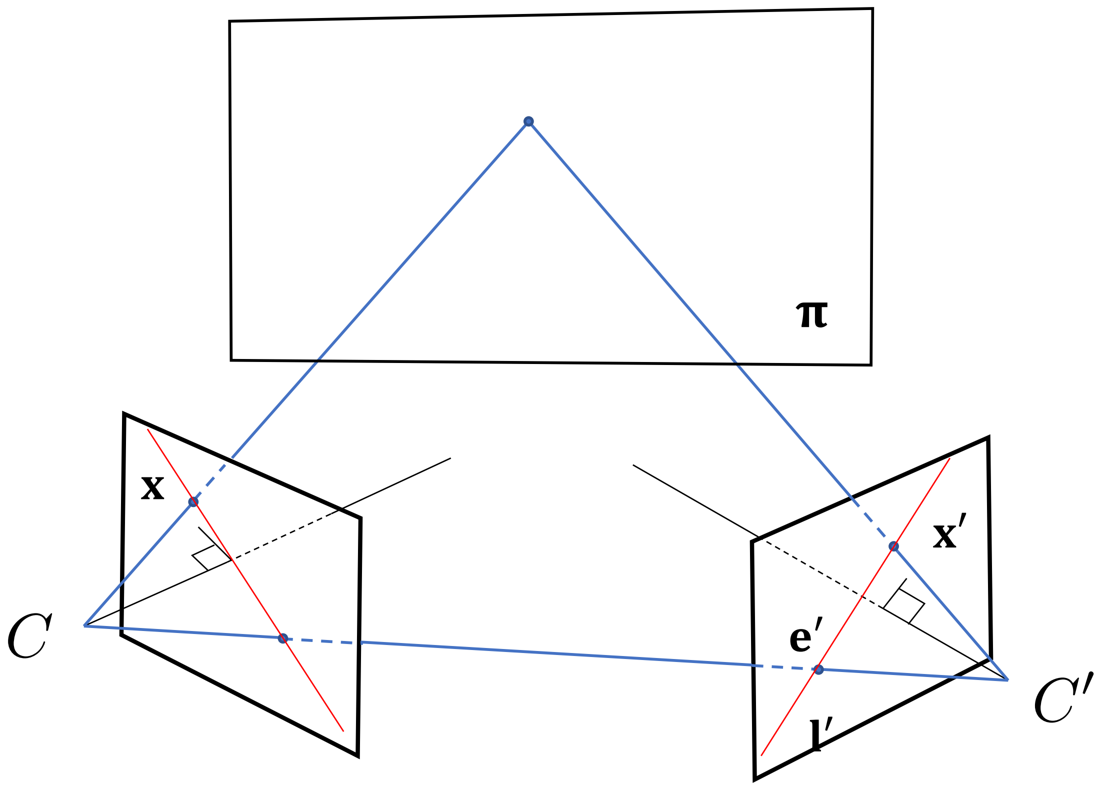
Then there is a 2D projective transformation $\mathbf{x}' \propto \mathtt{H}\mathbf{x}$

$\mathbf{l}'$ is given by $\mathbf{l}' = [\mathbf{e}']_\times \mathbf{x}'$

$$= [\mathbf{e}']_\times \mathtt{H}\mathbf{x}$$

$$= ([\mathbf{e}']_\times \mathtt{H})\mathbf{x}$$

$$\mathtt{F} = ([\mathbf{e}']_\times \mathtt{H})$$



A matrix representing vector cross product

- For a 3-vector $\mathbf{v}$, $[\mathbf{v}]_\times$ is defined to be a 3x3 matrix satisfying

$$\mathbf{v} \times \mathbf{x} = [\mathbf{v}]_\times \mathbf{x}$$

- It is given by

$$[\mathbf{v}]_\times = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$$

  - $[\mathbf{v}]_\times$ is a skew-symmetric matrix, i.e., $[\mathbf{v}]_\times^\top = -[\mathbf{v}]_\times$
  - $[\mathbf{v}]_\times$ has rank ≤ 2, because $[\mathbf{v}]_\times \mathbf{v} = \mathbf{0}$

Properties of fundamental matrix

- $\mathtt{F}^\top$ gives the relation for the reverse order of views:

$$\mathbf{l}' = \mathtt{F}\mathbf{x} \qquad \mathbf{l} = \mathtt{F}^\top \mathbf{x}'$$

- $\mathbf{x}'^\top \mathtt{F}\mathbf{x} = 0$, because $\mathbf{x}'^\top \mathbf{l}' = 0$
- $\mathtt{F}\mathbf{e} = \mathbf{0}$ and $\mathtt{F}^\top \mathbf{e}' = \mathbf{0}$, because any epipolar line $\mathbf{l}$ passes $\mathbf{e}$; thus $\mathbf{l}^\top \mathbf{e} = 0$, and $\mathbf{x}'^\top \mathtt{F}\mathbf{e} = 0$; this should hold for any $\mathbf{x}'$
- $\mathtt{F}$ has rank 2, because $\mathtt{F} = [\mathbf{e}']_\times \mathtt{H}$
    - Any matrix of rank 2 can be a fundamental matrix; proof omitted
- The DoF of $\mathtt{F}$ is seven; 3 x 3 – 1(scaling) – 1(rank=2) = 7
- $\mathtt{F}$ represents the geometric relation of a pair of uncalibrated cameras in a complete and concise manner

Deriving camera matrices from $\mathtt{F}$

- Proposition: Given a fundamental matrix $\mathtt{F}$ of two views, the camera matrices of the two views are given as

$$\mathtt{P} = \begin{bmatrix} \mathtt{I} & \mathbf{0} \end{bmatrix} \qquad \mathtt{P}' = \begin{bmatrix} \mathtt{SF} & \mathbf{e}' \end{bmatrix}$$

- where $\mathtt{S}$ is any skew-symmetric matrix and $\mathbf{e}'$ is the epipole:

$$\mathbf{e}'^{\top}\mathtt{F} = \mathbf{0}^{\top}$$

  - Remark: the above gives a projective reconstruction

- Lemma of the proposition: If $\mathtt{F}$ is a fundamental matrix of two views having camera matrices $\mathtt{P}$ and $\mathtt{P}'$, then $\mathtt{P}'^{\top}\mathtt{F}\mathtt{P}$ is skew-symmetric, and vice versa

Skew-symmetric: $\mathtt{A}^{\top} = -\mathtt{A}$

Deriving camera matrices from $\mathtt{F}$

- Proof of Lemma: If a matrix $\mathtt{A}$ is skew-symmetric, then it holds that for any $\mathbf{x}$, $\mathbf{x}^\top \mathtt{A}\mathbf{x} = 0$ and vice versa
  - Therefore, if $\mathtt{P}'^\top \mathtt{F}\mathtt{P}$ is skew-symmetric, $\mathbf{X}^\top \mathtt{P}'^\top \mathtt{F}\mathtt{P}\mathbf{X} = 0$ for any $\mathbf{X}$, and vice versa
  - We may set $\mathbf{x}' \propto \mathtt{P}'\mathbf{X}$ and $\mathbf{x} \propto \mathtt{P}\mathbf{X}$, resulting in $\mathbf{x}'^\top \mathtt{F}\mathbf{x} = 0$

- Proof of proposition: We only need to show $\mathtt{P}'^\top \mathtt{F}\mathtt{P}$ is skew-symmetric; this is done as follows

$$\mathtt{P}'^\top \mathtt{F}\mathtt{P} = \begin{bmatrix} \mathtt{S}\mathtt{F} & \mathbf{e}' \end{bmatrix}^\top \mathtt{F} \begin{bmatrix} \mathtt{I} & \mathbf{0} \end{bmatrix}$$
$$= \begin{bmatrix} -\mathtt{F}^\top \mathtt{S}\mathtt{F} & \mathbf{0} \\ \mathbf{e}'^\top \mathtt{F} & 0 \end{bmatrix} = \begin{bmatrix} -\mathtt{F}^\top \mathtt{S}\mathtt{F} & \mathbf{0} \\ \mathbf{0}^\top & 0 \end{bmatrix}$$

Essential matrix

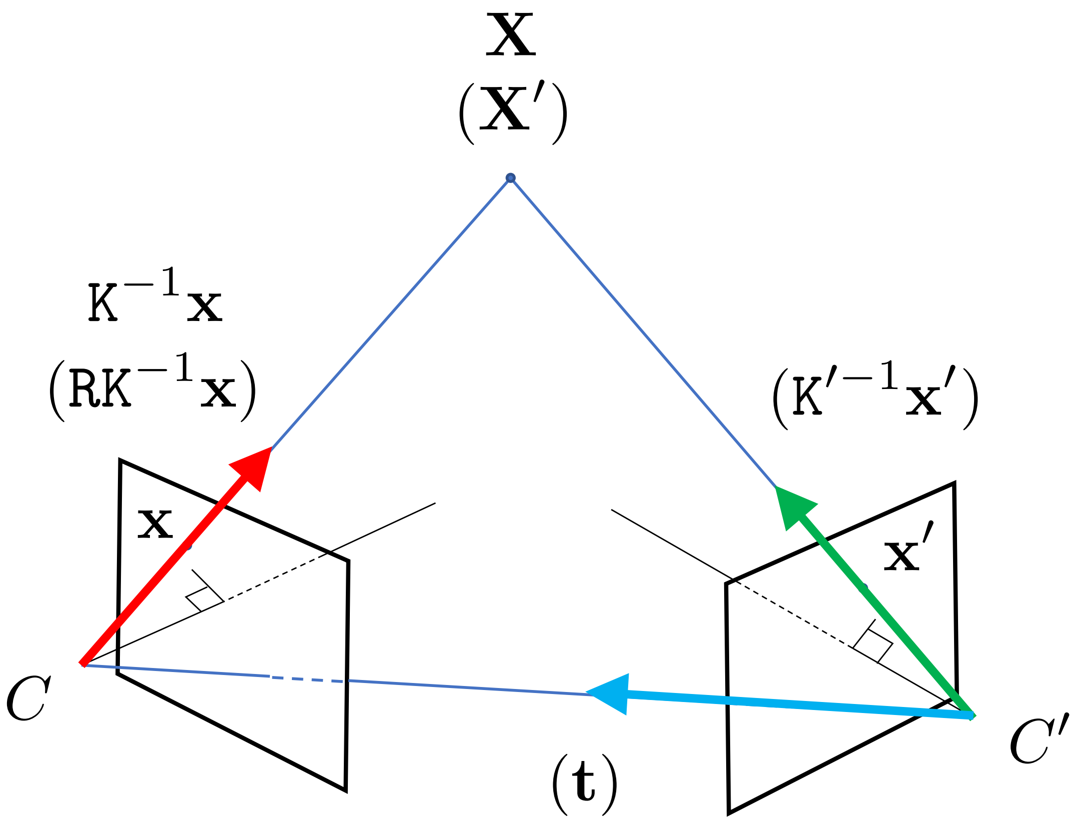
- $\mathtt{E} \equiv \mathtt{K}'^{\top}\mathtt{F}\mathtt{K}$ is called the essential matrix
- $\mathtt{E}$ gives a two-view relation similar to $\mathtt{F}$ *when the camera(s) are calibrated*
  - Substituting $\mathbf{x} \propto \mathtt{K}\tilde{\mathbf{X}} = \mathtt{K}\begin{bmatrix} X & Y & Z \end{bmatrix}^{\top}$ and $\mathbf{x}' \propto \mathtt{K}'\tilde{\mathbf{X}}' = \mathtt{K}'\begin{bmatrix} X' & Y' & Z' \end{bmatrix}^{\top}$ into $\mathbf{x}'^{\top}\mathtt{F}\mathbf{x} = 0$, we have

$$\mathbf{x}'^{\top}\mathtt{F}\mathbf{x} = \tilde{\mathbf{X}}'^{\top}\mathtt{K}'^{\top}\mathtt{F}\mathtt{K}\tilde{\mathbf{X}} = \tilde{\mathbf{X}}'^{\top}(\mathtt{K}'^{\top}\mathtt{F}\mathtt{K})\tilde{\mathbf{X}} = 0$$

- Properties:
  - Denote the coordinate trans. between the two camera coord. by $\tilde{\mathbf{X}}' = \mathtt{R}\tilde{\mathbf{X}} + \mathbf{t}$; then $\mathtt{E} = [\mathbf{t}]_{\times}\mathtt{R}$
  - DoF of $\mathtt{E}$ is five (rotation + translation – scaling)
  - A 3x3 matrix $\mathtt{E}$ is an essential matrix if and only if the two of its singular values are identical and the rest is zero; proof omitted

Essential matrix

- Proof of $\mathtt{E} = [\mathbf{t}]_\times \mathtt{R}$



Coordinate trans. from 1st to 2nd camera:

$$\tilde{\mathbf{X}}' = \mathtt{R}\tilde{\mathbf{X}} + \mathbf{t}$$

The three colored vectors lie on a plane:

$$(\mathtt{K}'^{-1}\mathbf{x}')^\top(\mathbf{t} \times \mathtt{R}\mathtt{K}^{-1}\mathbf{x}) = 0$$

$$\mathbf{x}'^\top \mathtt{K}'^{-\top}\mathbf{t} \times \mathtt{R}\mathtt{K}^{-1}\mathbf{x} = 0$$

$$\mathbf{x}'^\top \mathtt{K}'^{-\top}\underbrace{[\mathbf{t}]_\times \mathtt{R}}_{\mathtt{E} = [\mathbf{t}]_\times \mathtt{R}}\mathtt{K}^{-1}\mathbf{x} = 0$$

Essential matrix

- If $\mathtt{E}$ is an essential matrix, then its singular values is $[s, s, 0]$
  - $\mathtt{E}$ is of rank 2 just like $\mathtt{F}$
- Given $\mathtt{E}$, we can obtain $\mathbf{t}$ and $\mathtt{R}$ as shown below
- The SVD of $\mathtt{E}$ is given as

$$\mathtt{E} = \mathtt{U} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} \mathtt{V}^\top$$

- Suppose the camera matrix of 1st view to be $\mathtt{P} = \mathtt{K} \begin{bmatrix} \mathtt{I} & \mathbf{0} \end{bmatrix}$
- Then, 2nd camera matrix should be one of the four matrices:

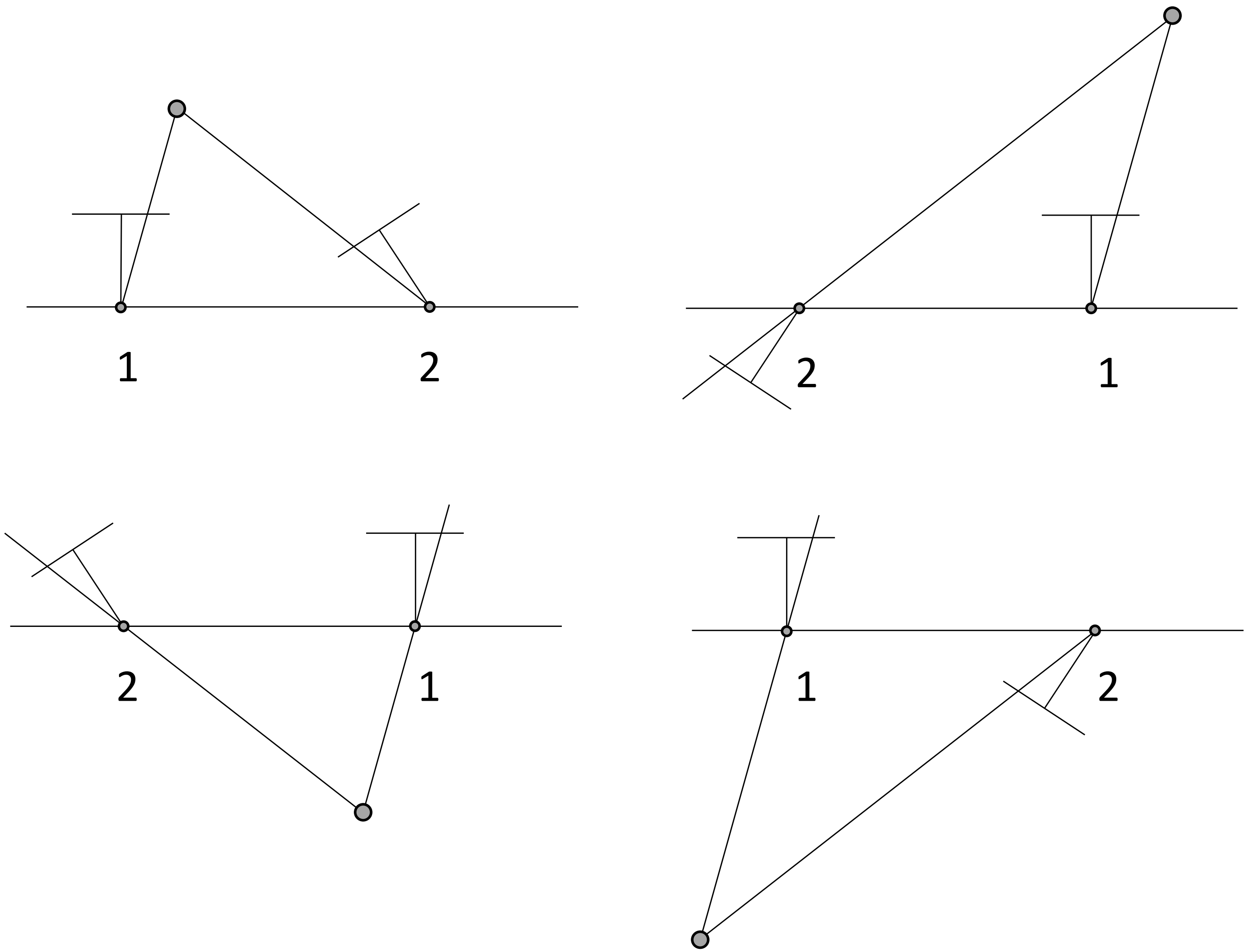
$$\mathtt{P}' = \mathtt{K}' \begin{bmatrix} \mathtt{UWV}^\top & \mathbf{u}_3 \end{bmatrix}$$
$$\mathtt{P}' = \mathtt{K}' \begin{bmatrix} \mathtt{UWV}^\top & -\mathbf{u}_3 \end{bmatrix}$$
$$\mathtt{P}' = \mathtt{K}' \begin{bmatrix} \mathtt{UW}^\top\mathtt{V}^\top & \mathbf{u}_3 \end{bmatrix}$$
$$\mathtt{P}' = \mathtt{K} \begin{bmatrix} \mathtt{UW}^\top\mathtt{V}^\top & -\mathbf{u}_3 \end{bmatrix}$$

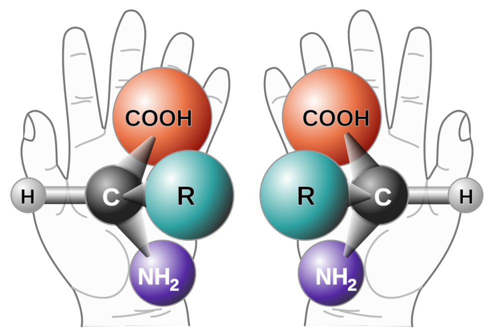
where

$$\mathtt{W} \equiv \begin{bmatrix} & -1 & \\ 1 & & \\ & & 1 \end{bmatrix} \qquad \mathbf{u}_3 = \mathtt{U} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Obtaining R and t from E

- Four solutions and relative camera poses
  - A single solution is physically possible: 3D points will be in front of both the cameras for only one case (“chirality check”)





chirality

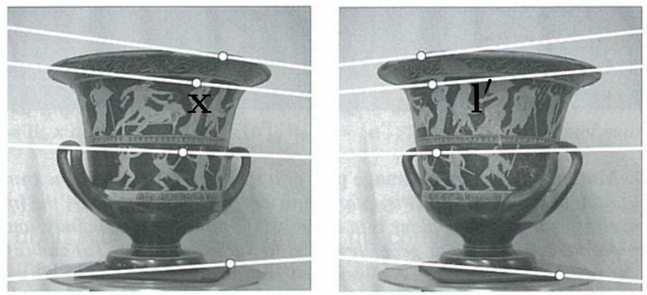
Application of epipolar (two-view) geometry

- Finding point correspondences between two images
  - Starting from $\mathtt{F}$ (seven point matches) or $\mathtt{E}$ (five)
  - Once epipolar geometry is obtained, you need only to search along the epipolar line for the corresponding point
  - Robust correspondence estimation: RANSAC etc.

- Reconstructing 3D structure or camera poses
  - Projective reconstruction from $\mathtt{F}$

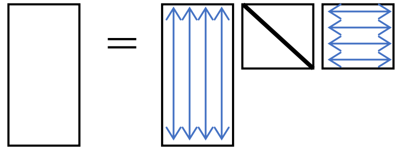
$$\mathtt{P} = \begin{bmatrix}\mathtt{I} & \mathbf{0}\end{bmatrix} \qquad \mathtt{P}' = \begin{bmatrix}\mathtt{SF} & \mathbf{e}'\end{bmatrix}$$

  - Similarity reconstruction from $\mathtt{E}$

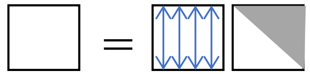


Summary: matrix decompositions

- SVD; singular value decomposition
  - Any $m \times n$ matrix $\mathtt{A}$ can be decomposed as $\mathtt{A} = \mathtt{UWV}^\top$
  - $\mathtt{U}$ and $\mathtt{V}$ are orthogonal: $\mathtt{U}^\top\mathtt{U} = \mathtt{I}$ $\mathtt{V}^\top\mathtt{V} = \mathtt{I}$
  - $\mathtt{W}$ is diagonal whose diagonal entries are called singular values
  - Unique if singular values are sorted in descending order



- QR decomposition
  - Any $m \times m$ matrix $\mathtt{A}$ can be decomposed as $\mathtt{A} = \mathtt{QR}$
  - $\mathtt{Q}$ is orthogonal ($\mathtt{Q}^\top\mathtt{Q} = \mathtt{I}$) and $\mathtt{R}$ is an upper-right triangular matrix
  - Decomposition is unique



- Cholesky decomposition
  - Any $m \times m$ positive definite symmetric matrix $\mathtt{A}$ can be decomposed as $\mathtt{A} = \mathtt{LL}^\top$
  - L is a lower-left triangular matrix