1. Projective geometry

- Homogeneous representation of points and lines in 2D space
- 2D projective space
- Points at infinity and the line at infinity
- Conics and dual conics
- Projective transformation
- Hierarchy of 2D projective transformation

Notation

- Scalars: $x, y, \ldots$
- Vectors: $\mathbf{x}, \mathbf{y}, \ldots$
  - Transpose: $\mathbf{x}^\top, \mathbf{y}^\top, \ldots$
- Matrices: $\mathtt{A}, \mathtt{B}, \mathtt{P}, \ldots$
  - Transpose: $\mathtt{A}^\top, \mathtt{B}^\top, \mathtt{P}^\top, \ldots$

Homogeneous representation of lines in 2D space

- A line on a xy plane: $ax + by + c = 0$
- Any line can be represented by a 3-vector $\mathbf{l} = [a, b, c]^\top$

- The same line is given by $k(a, b, c)^\top = (ka, kb, kc)^\top$

$$kax + kby + kc = k(ax + by + c) = 0$$

- In this sense, $(a, b, c)^\top$ is equivalent to $k(a, b, c)^\top$

- Such vectors are called homogeneous vectors
- This equivalence will be represented by $\propto$
  - Two vectors on both sides are parallel

$$(a, b, c)^\top \propto k(a, b, c)^\top$$

Homogeneous representation of points in 2D space

- A point (x, y) is on a line $\mathbf{l} = (a, b, c)^\top$

$$\Leftrightarrow \quad \mathbf{l}^\top \mathbf{x} = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = ax + by + c = 0$$

- Homogeneous representation of a point: $\mathbf{x} = (x, y, 1)^\top$
  - The same point is given by: $k(x, y, 1)^\top = (kx, ky, k)^\top$

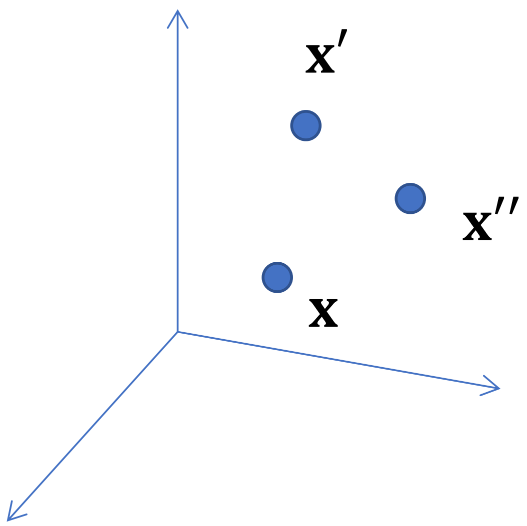
$$(x, y, 1)^\top \propto k(x, y, 1)^\top$$

- To recover the original (inhomogeneous) coordinates from homogeneous coordinates, simply do

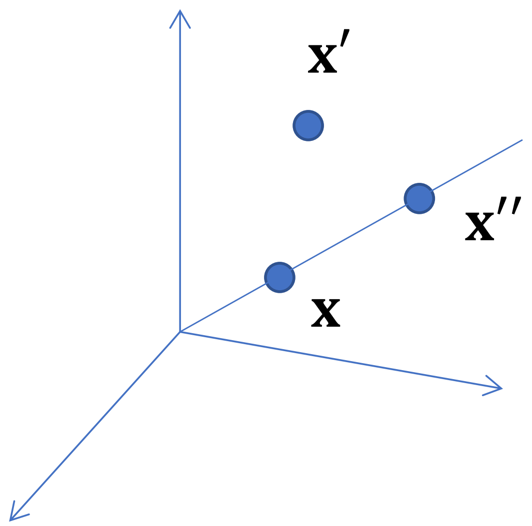
$$\mathbf{x} = (x_1, x_2, x_3)^\top \quad \Rightarrow \quad \left(\frac{x_1}{x_3}, \frac{x_2}{x_3} \right)$$

2D projective space

- Sets of equivalent vectors in this sense form a special space called the projective space



3D Euclidean space



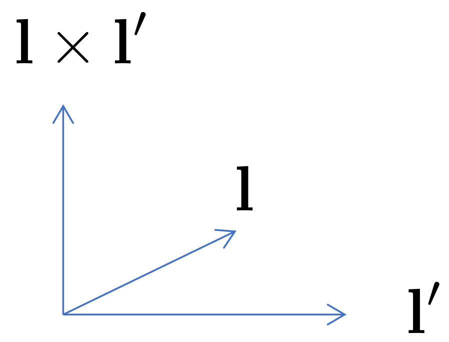
2D projective space

Intersection of lines

- Q) What is the crossing point of lines? $\mathbf{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \mathbf{l}' = \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix}$

- A) It is given by their cross product:

$$\mathbf{x} \equiv \mathbf{l} \times \mathbf{l}'$$



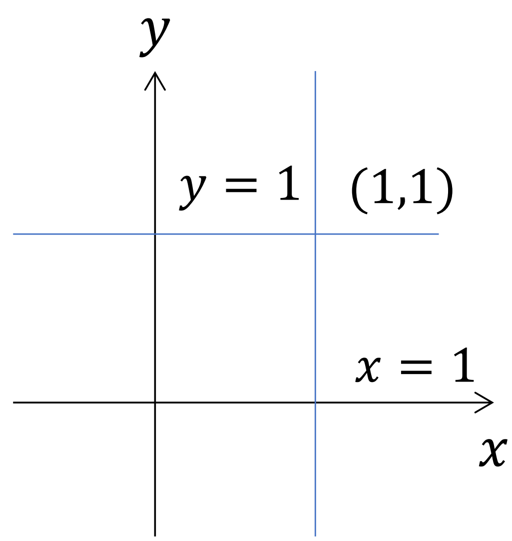
- Proof: $\mathbf{l}^\top \mathbf{x} = \mathbf{l}^\top (\mathbf{l} \times \mathbf{l}') = 0$
  (Scalar triple product)

- E.g. Crossing point of $x = 1$ and $y = 1$?

$$\mathbf{l} = [-1, 0, 1]^\top \quad \mathbf{l}' = [0, -1, 1]^\top$$

$$\mathbf{x} = \mathbf{l} \times \mathbf{l}' = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad (1/1, 1/1) = (1, 1)$$



Points at infinity

- Consider the intersection of $\mathbf{l} = (a, b, c)^\top$ and $\mathbf{l}' = (a, b, c')^\top$

$$\mathbf{l} \times \mathbf{l}' = (bc' - bc, ca - ac', ab - ab)^\top = (c' - c)(b, -a, 0)^\top$$

  - Conversion to inhomogeneous coordinates$\cdots$

$$(b/0, -a/0) = (\infty, \infty)$$

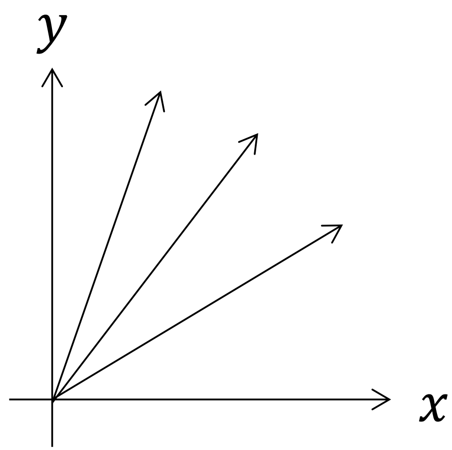
- A point with homogeneous coordinates $(x, y, 0)^\top$ is called "a point at infinity"

  - A point $(x_1, x_2, x_3)^\top$ gives a finite point if $x_3 \neq 0$ and gives a point at infinity if $x_3 = 0$

Points at infinity

- In inhomogeneous domain, parallel lines do not have an intersection
- In homogenous domain, parallel lines have an intersection at a point at infinity
- Points at infinity and finite points can be treated equally ← an advantage of using homogeneous representation
- Note that there are an infinite number of 'points at infinity'

$$(a, b, 0)^\top$$



The line at infinity

- The special line $\mathbf{l} = (0, 0, 1)^\top$ is called "the line at infinity"

- The name comes from the fact that every point at infinity will lie on this line

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = 0$$

- This line is unique, and will be denoted by

$$\mathbf{l}_\infty = (0, 0, 1)^\top$$

Duality of points and lines

- There is symmetry between points and lines

$$\mathbf{l}^\top \mathbf{x} = \mathbf{x}^\top \mathbf{l} = 0$$

- Exchange of points and lines in a proposition won't change its correctness
  - E.g. Intersection of lines is given by

$$\mathbf{x} = \mathbf{l} \times \mathbf{l}'$$

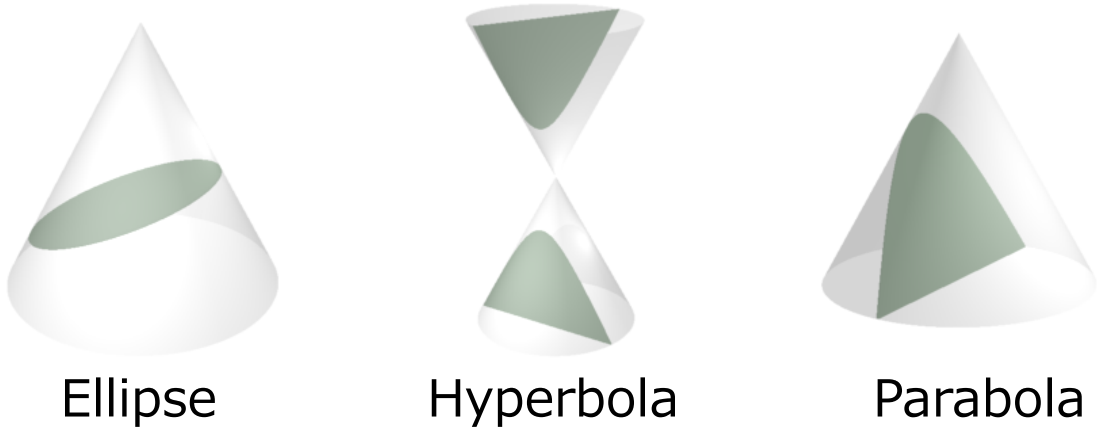
$\Longleftrightarrow$ A line passing through two points is given by

$$\mathbf{l} = \mathbf{x} \times \mathbf{x}'$$

- Prove this directly without using the duality (1st assignment)

Conics

- A conic = a curve obtained as the intersection of a cone with a plane



Any conic on a xy plane is given by

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

$$\Downarrow \quad x \leftarrow x_1/x_3 \;\text{ and }\; y \leftarrow x_2/x_3$$

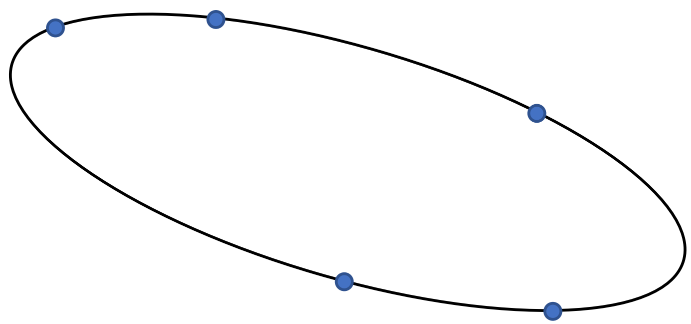
$$ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

Conics

- Or, can also be written as $\mathbf{x}^\top \mathtt{C}\mathbf{x} = 0$

$$\mathtt{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

- Any 3×3 symmetric matrix represents a conic
  - Matrices scaled by constants give the same conic
- Specifying five points determines a conic (passing these points)



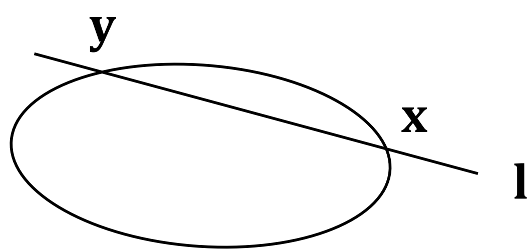
Tangents to a conic

- The line $\mathbf{l}$ tangent to a conic $\mathtt{C}$ at a point $\mathbf{x}$ is given by

$$\mathbf{l} = \mathtt{C}\mathbf{x}$$

Proof: $\mathbf{x}$ lies on $\mathbf{l}$, since $\mathbf{x}^\top \mathbf{l} = \mathbf{x}^\top \mathtt{C}\mathbf{x} = 0$

Assume that there exists another point $\mathbf{y}$ lying on $\mathtt{C}$ and $\mathbf{l}$.



Then, for any α, it should hold that

$$(\mathbf{x} + \alpha\mathbf{y})^\top \mathtt{C}(\mathbf{x} + \alpha\mathbf{y}) = 0$$

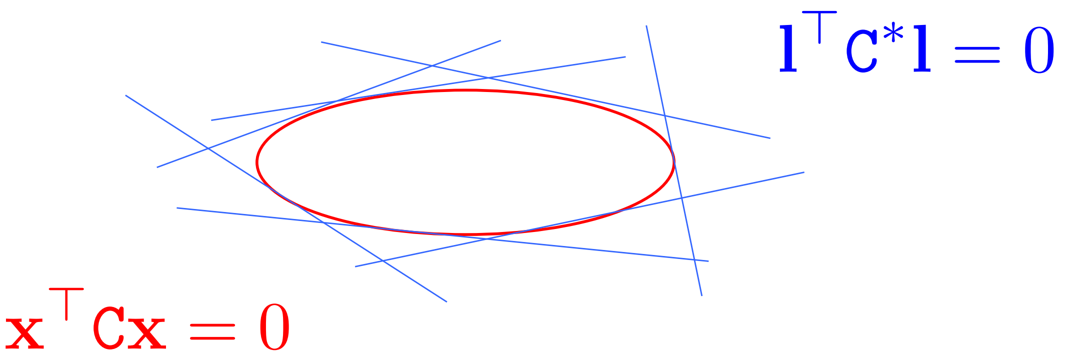
This means that any point $\mathbf{x} + \alpha\mathbf{y}$ should lie on $\mathtt{C}$.

This is possible only if $\mathtt{C}$ is a line or there exists no such point $\mathbf{y}$.

Dual conics

- A conic $\mathtt{C}$ gives a set of points. A dual conic $\mathtt{C}^*$ is another type of conics, which gives a set of lines
  - Also known as line conics

- Consider $\mathbf{l}^\top \mathtt{C}^* \mathbf{l} = 0$, where $\mathtt{C}^* \equiv \mathtt{C}^{-1}$
- This gives a set of tangent lines to conic $\mathtt{C}$ ---- (*)



Show the above (*) is true (2nd assignment).

2D projective transformation

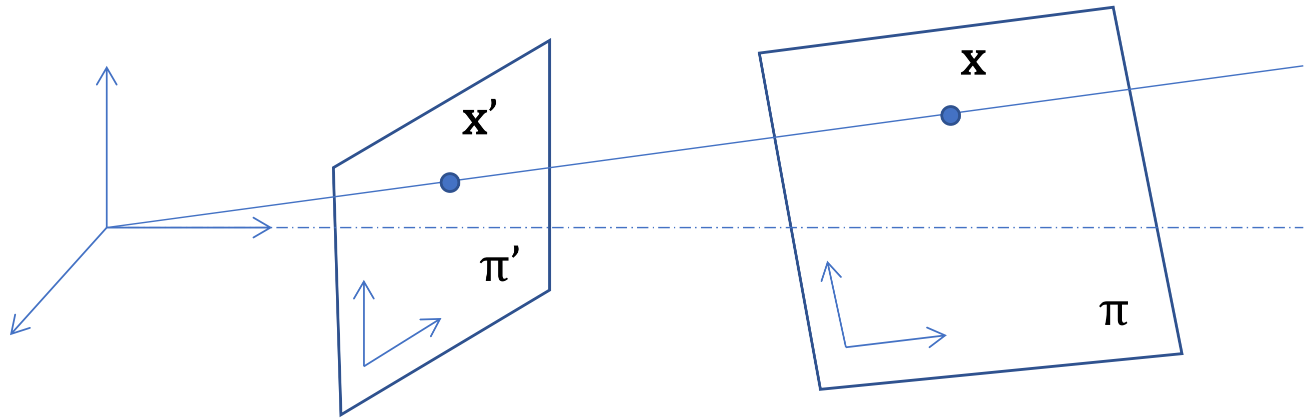
- Projective transformation (different names: *projectivity, homography, colineation*)
  - Definition: *An invertible mapping from/to 2D projective spaces that satisfies that if three points lie on a line*, then the mapped points lie on a line.
  - Intuitively, any mapping from a plane to a plane that maps a line to a line

- Such a transformation ($\mathbf{x} \rightarrow \mathbf{x}'$) is given by the following equation:

$$\mathbf{x}' \propto \mathtt{H}\mathbf{x} \quad \text{or} \quad \mathbf{x}' \propto \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \mathbf{x}$$

An example: central projection

- Central projection: a point on a plane is mapped onto a point on another plane as shown below



- Obviously, central projection maps a line to a line, and thus it is a 2d projective transformation

Calculating a projective transformation

- If you want to represent the points before/after the mapping in inhomogeneous coordinates⋯

$$\mathbf{x}' \propto \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \mathbf{x}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \propto \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11}x + h_{12}y + h_{13} \\ h_{21}x + h_{22}y + h_{23} \\ h_{31}x + h_{32}y + h_{33} \end{bmatrix}$$

$$x' = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}}$$

$$y' = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

Transformation of lines and conics

- A 2D projective transformation mapping a point as

$$\mathbf{x}' \propto \mathtt{H}\mathbf{x}$$

- maps a line as

$$\mathbf{l}' \propto \mathtt{H}^{-\top}\mathbf{l}$$

  - proof:

$$\mathbf{x}'^{\top}\mathbf{l}' = (\mathbf{x}^{\top}\mathtt{H}^{\top})\mathbf{l}' = (\mathbf{x}^{\top}\mathtt{H}^{\top})(\mathtt{H}^{-\top}\mathbf{l}) = \mathbf{x}^{\top}\mathbf{l} = 0$$

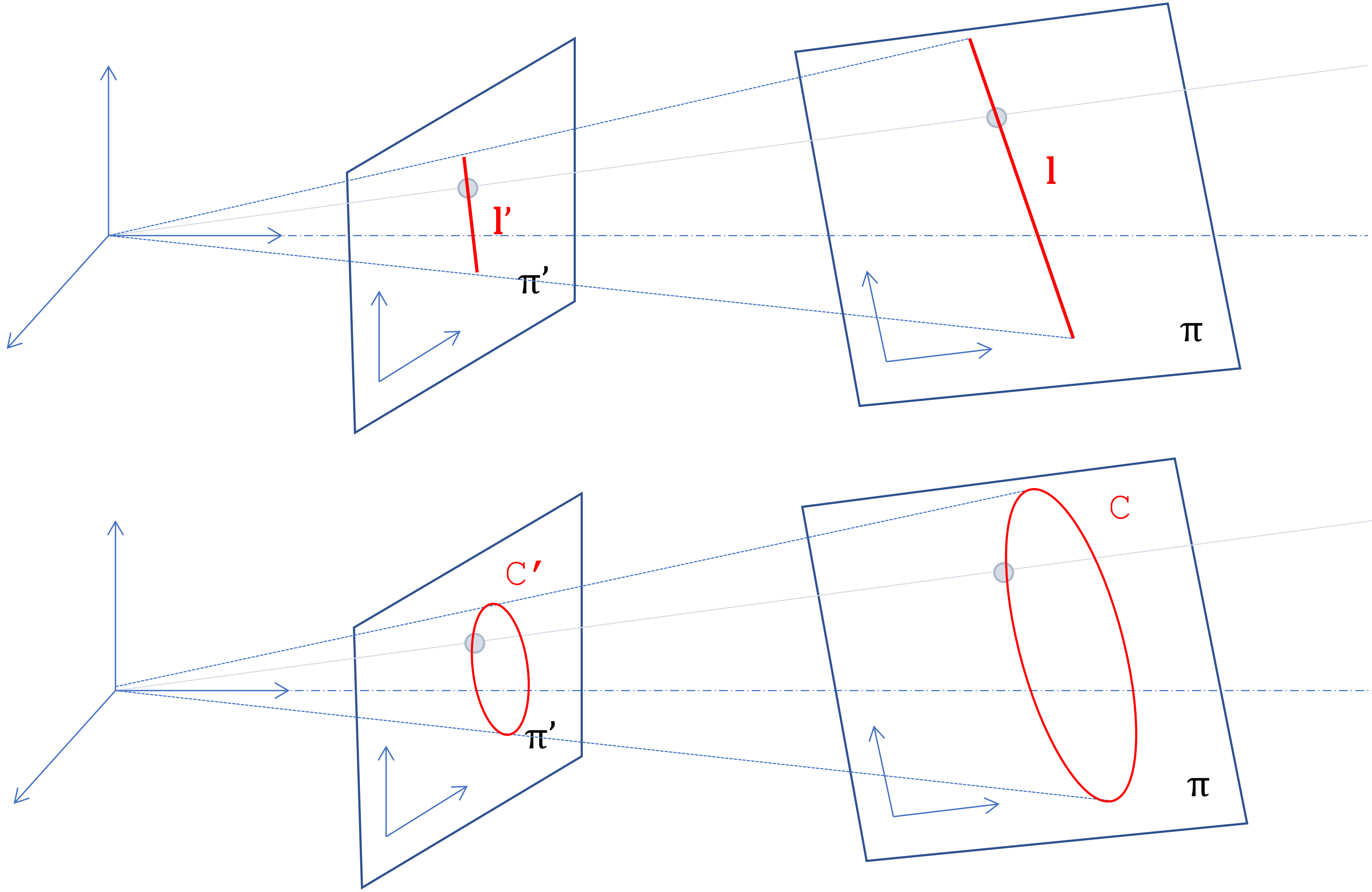
- and maps a conic as

$$\mathtt{C}' \propto \mathtt{H}^{-\top}\mathtt{C}\mathtt{H}^{-1}$$

  - proof:

$$\mathbf{x}'^{\top}\mathtt{C}'\mathbf{x}' = \mathbf{x}^{\top}\mathtt{H}^{\top}\mathtt{C}'\mathtt{H}\mathbf{x} = \mathbf{x}^{\top}\mathtt{H}^{\top}\mathtt{H}^{-\top}\mathtt{C}\mathtt{H}^{-1}\mathtt{H}\mathbf{x} = \mathbf{x}^{\top}\mathtt{C}\mathbf{x} = 0$$

Transformation of lines and conics



Hierarchy of 2D projective transformation

- Projective transformation can be classified into the following four types in the order of increasing degrees of freedom:

1. Euclidean transformation / Isometry
2. Similarity transformation
3. Affine transformation
4. Projective transformation (full-projective ---)

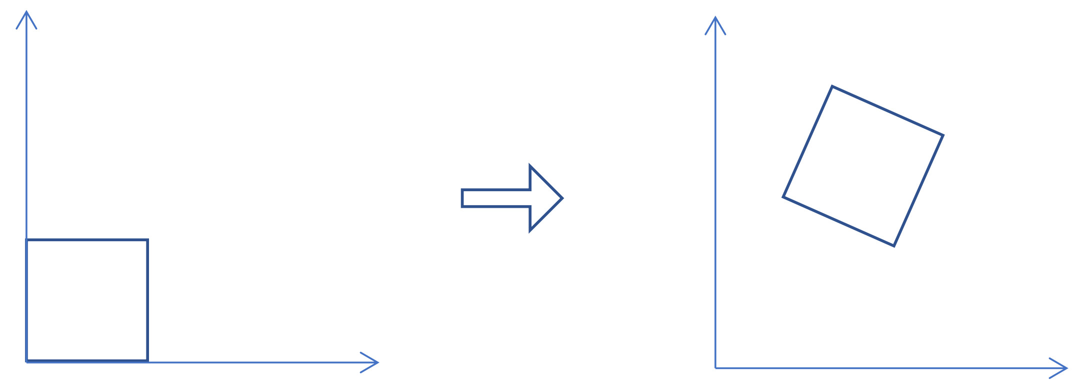
$$\mathbf{x}' \propto \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \mathbf{x}$$

Euclidean transformation

- 2D projective transformations given as follows:

$$\mathbf{x}' \propto \begin{bmatrix} \varepsilon\cos\theta & -\sin\theta & t_x \\ \varepsilon\sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}$$

  - Called Euclidean trans. if $\varepsilon=1$; isometry if $\varepsilon=-1$ or 1.
  - Combination of 2D rotation and translation
  - Length and area are preserved (invariant)

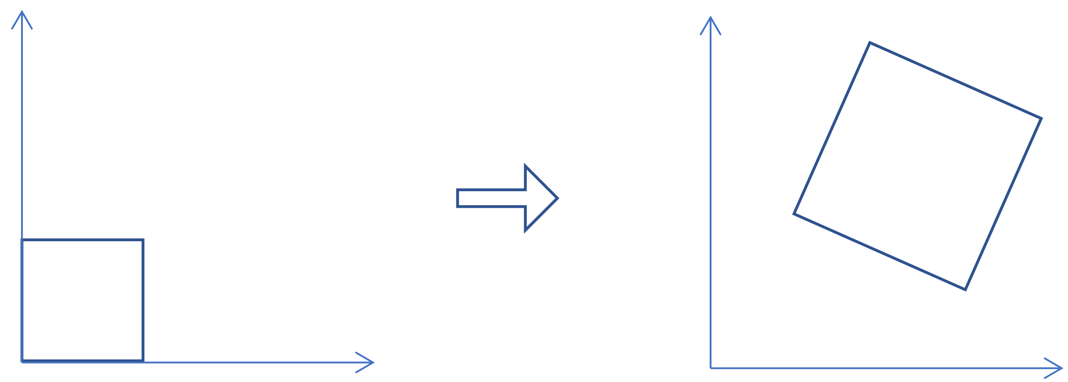


Similarity transformation

- 2D projective transformations given as follows:

$$\mathbf{x}' \propto \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}$$

  - Combination of 2D rotation, translation, and scaling
  - Shape and angle are preserved (invariant)

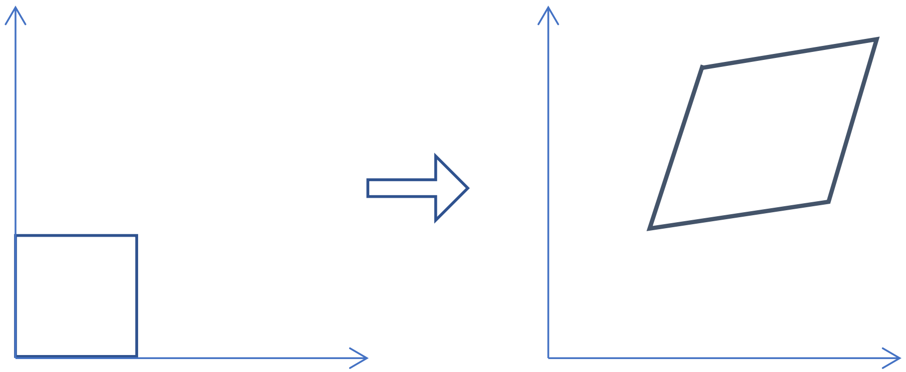


Affine transformation

- 2D projective transformations given as follows:

$$\mathbf{x}' \propto \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}$$

  - Parallelism is preserved
  - Points at infinity are mapped to points at infinity



$$\mathbf{x}' \propto \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$

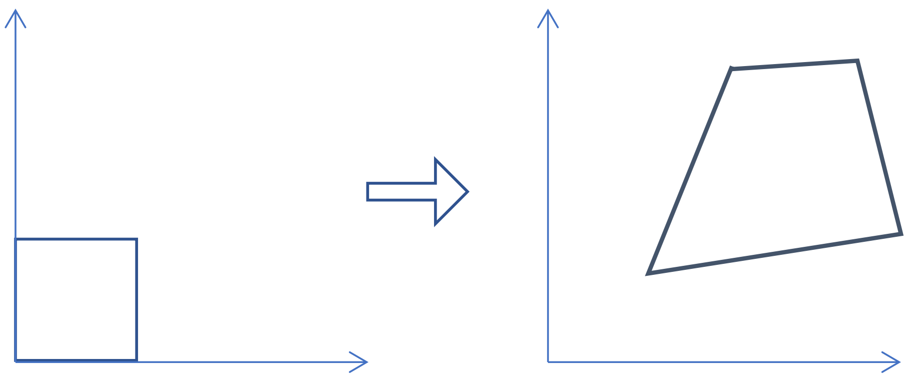
$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \\ 0 \end{bmatrix}$$

Projective transformation

- The most general one:

$$\mathbf{x}' \propto \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \mathbf{x}$$

  - Colinearity and cross-ratio are preserved
  - Points at infinity can be mapped to finite points



$$\mathbf{x}' \propto \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} h_{11}x_1 + h_{12}x_2 \\ h_{21}x_1 + h_{22}x_2 \\ h_{31}x_1 + h_{32}x_2 \end{bmatrix}$$

Transformations and images

- Images of a plane created by different transformations


similarity

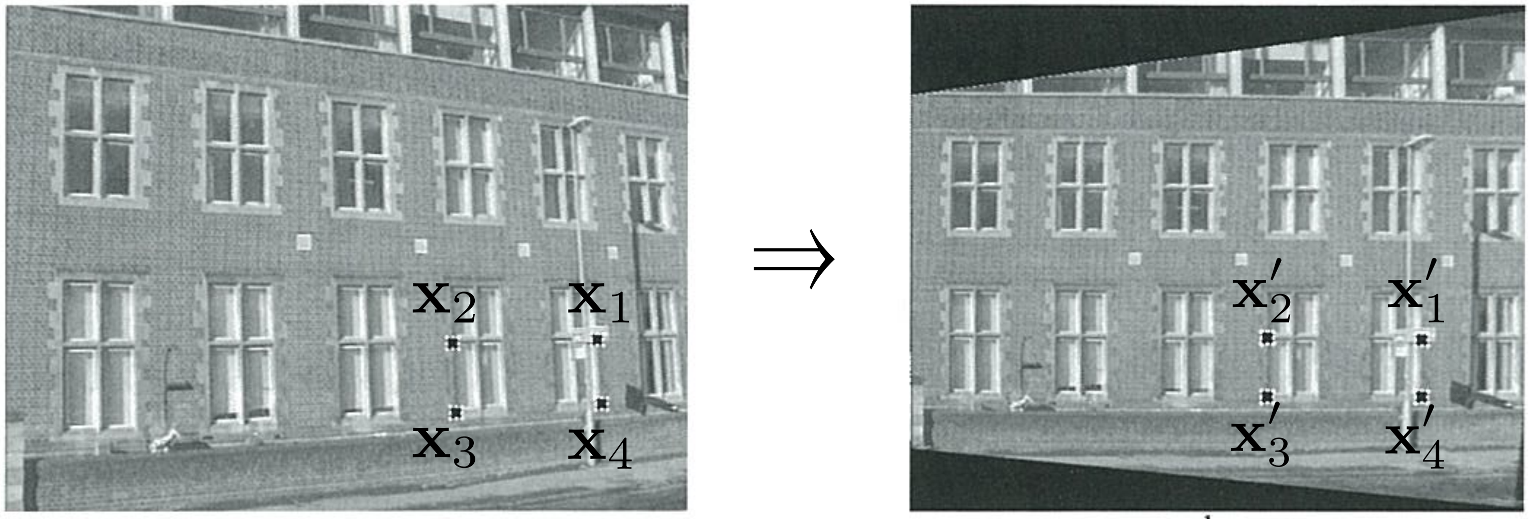

affine


projective

[Hartley-Zisserman03]

Image rectification

- Given a projective transform of a plane, we want to find a transformation that maps it onto a similarity transform of the same plane



[Hartley-Zisserman03]

- Any projective trans. is determined by four point pairs